1. [12 bodov]
Riešte sústavu, riešenie napíšte v algebraickom tvare, urobte skúšku správnosti.

$$(1-i)x - 2iy = 3$$
$$(2+i)x + (1+i)y = 1+i$$

Riešenie pomocou Cramerovho pravidla

$$d = \begin{vmatrix} 1-i & -2i \\ 2+i & 1+i \end{vmatrix} = (1-i)(1+i) + 2i(2+i) = 2 + 4i - 2 = 4i\ ,$$
$$d_1 = \begin{vmatrix} 3 & -2i \\ 1+i & 1+i \end{vmatrix} = 3 + 3i + 2i(1+i) = 1 + 5i\ ,$$
$$d_2 = \begin{vmatrix} 1-i & 3 \\ 2+i & 1+i \end{vmatrix} = 2 - 6 - 3i = -4 - 3i\ ,$$
$$x = \tfrac{1+5i}{4i}\,\tfrac{-i}{-i} = \tfrac{5-i}{4}$$
$$y = \tfrac{-4-3i}{4i}\,\tfrac{-i}{-i} = \tfrac{4i-3}{4} \qquad P = \left\{\left(\tfrac{5}{4} - \tfrac{1}{4}i\,,\ -\tfrac{3}{4} + i\right)\right\}$$
$$L_1 = (1-i)\,\tfrac{5-i}{4} - 2i\,\tfrac{4i-3}{4} = \tfrac{1}{4}[5 - i - 5i - 1 + 8 + 6i] = 3 = P_1$$
$$L_2 = (2+i)\,\tfrac{5-i}{4} + (1+i)\,\tfrac{4i-3}{4} = \tfrac{1}{4}[10 - 2i + 5i + 1 + 4i - 3 - 4 - 3i] = 1 + i = P_2$$

2. [18bodov] Rozšírenú maticu sústavy upravte na redukovanú stupňovitú a napíšte množinu všetkých jej riešení. Urobte skúšku správnosti.

$$2x_1 - 5x_2 + 3x_3 - x_4 = 1$$
$$x_1 - 2x_2 + x_3 + 2x_4 = 0$$
$$3x_1 - 4x_2 + x_3 + 2x_4 = 2$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 2 & -5 & 3 & -1 & 1 \\ 3 & -4 & 1 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 0 & -1 & 1 & -5 & 1 \\ 0 & 2 & -2 & -4 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 0 & -1 & 1 & -5 & 1 \\ 0 & 0 & 0 & -14 & 4 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 0 & -1 & 1 & -5 & 1 \\ 0 & 0 & 0 & 1 & -2/7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 4/7 \\ 0 & -1 & 1 & 0 & -3/7 \\ 0 & 0 & 0 & 1 & -2/7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 10/7 \\ 0 & 1 & -1 & 0 & 3/7 \\ 0 & 0 & 0 & 1 & -2/7 \end{array}\right)$$

$$P = \left\{\left(\tfrac{10}{7} + a\,,\ \tfrac{3}{7} + a\,,\ a\,,\ -\tfrac{2}{7}\right); a \in \mathbb{R}\right\}$$
$$L_1 = 2(\tfrac{10}{7} + a) - 5(\tfrac{3}{7} + a) + 3a + \tfrac{2}{7} = 1 = P_1$$
$$L_2 = \tfrac{10}{7} + a - 2(\tfrac{3}{7} + a) + a - \tfrac{4}{7} = 0 = P_2$$
$$L_3 = 3(\tfrac{10}{7} + a) - 4(\tfrac{3}{7} + a) + a - \tfrac{4}{7} = 2 = P_3$$

3. [15 bodov] Dané sú matice

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Určte hodnosť (rank) matice A a matice B, a vypočítajte matice
$A+B,\ B+A,\quad AB,\ (AB)^{-1},\quad BA,\ (BA)^{-1}$

Riešenie

$\operatorname{rank} A = 2,\ B \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \implies \operatorname{rank} B = 2$

$A+B$ ani $B+A$ nie je definované

$AB = \begin{pmatrix} 7 & 3 \\ -1 & 0 \end{pmatrix} \qquad \det(AB) = 3, \quad (AB)^{-1} = \frac{1}{3}\begin{pmatrix} 0 & -3 \\ 1 & 7 \end{pmatrix}$

$BA = \begin{pmatrix} 3 & 6 & 3 \\ 1 & 3 & 0 \\ 2 & 5 & 1 \end{pmatrix}$

$\det(BA) = 3\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 5 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{vmatrix} = 0 \implies \nexists (BA)^{-1}.$

Pr.4

Polynóm $f(x) = 4x^5 - 10x^4 - 4x^3 + 9x^2 + 12x + 4$ napíšte ako súčin ireducibilných polynómov nad $\mathbb{R}$ aj nad $\mathbb{C}$.
$c \in \left\{\pm 1, \pm 2, \pm 4, \pm\frac{1}{2}, \pm\frac{1}{4}\right\}$

$f(1) = 15,\ f(-1) = -9$

	4	−10	−4	9	12	4	
2	4	−2	−8	−7	−2	0	
2	4	6	4	1	0		

$f(x) = (x-2)^2(4x^3 + 6x^2 + 4x + 1)$

	4	6	4	1
$-\frac{1}{2}$	4	4	2	0

$D = 16 - 32 = -16 < 0 \implies f(x) = \underbrace{4(x-2)^2(x+\tfrac{1}{2})(x^2+x+\tfrac{1}{2})}_{nad\ \mathbb{R}}$

$x_{1,2} = \frac{-4\pm 4i}{8} = -\frac{1}{2} \pm \frac{1}{2}i, \quad f(x) = \underbrace{4(x-2)^2(x+\tfrac{1}{2})(x+\tfrac{1}{2}+\tfrac{1}{2}i)(x+\tfrac{1}{2}-\tfrac{1}{2}i)}_{nad\ \mathbb{C}}$

Pr5

Funkciu $f(x) = \dfrac{2x^2+4x+3}{(x^2+3x+2)(x^2+x+1)}$

napíšte ako súčet elementárnych zlomkov nad $\mathbb{R}$

$(x^2+3x+2) = (x+2)(x+1), \qquad D = 1 - 4 = -3 < 0$

$$f(x) = \frac{a}{x+1} + \frac{b}{x+2} + \frac{cx+d}{x^2+x+1}$$

$2x^2+4x+3 = a(x+2)(x^2+x+1) + b(x+1)(x^2+x+1) + (cx+d)(x+2)(x+1)$

$x=-1$: $2-4+3 = 1 = a\cdot(-1+2) \qquad \underline{a=1}$

$x=-2$: $8-8+3 = 3 = (-2+1)(4-2+1)b = -3b \quad \underline{b=-1}$

$x=0$: $3 = 2a+b+2d = 2-1+2d \implies \underline{d=1}$

$x=1$: $9 = 9a+6b+6c+6d = 9+6c \implies \underline{c=0}$

$$\underline{f(x) = \frac{1}{x+1} + \frac{-1}{x+2} + \frac{1}{x^2+x+1}}$$

Pr.6

Daný je bod $A = [1,1,1]$ a priamky

$$p \equiv x = 3 + t \qquad\qquad q \equiv x = 3 - 2t$$
$$y = 2 + 2t \qquad ; \qquad y = -4 + 2t$$
$$z = 5 + 2t\,,\ t \in \mathbb{R} \qquad\qquad z = t\,,\ t \in \mathbb{R}.$$

a) Zistite, či sú dané priamky rôznobežné a určte $p \cap q$.
b) Vypočítajte vzdialenosť $\mathrm{dist}(A,p)$ bodu A od priamky p.
c) Určte všeobecnú rovnicu roviny ρ, v ktorej ležia obe dané priamky.
d) Vypočítajte vzdialenosť $\mathrm{dist}(A,\rho)$

Riešenie

a) $P = (3,2,5)$, $Q = (3,-4,0)$, $\mathbf{u} = (1,2,2) \parallel p$, $\mathbf{v} = (-2,2,1) \parallel q$

$$(P-Q)\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} 0 & 6 & 5 \\ 1 & 2 & 2 \\ -2 & 2 & 1 \end{vmatrix} = 0\cdot(-2) - 6\cdot 5 + 5\cdot 6 = 0 \implies \text{sú rôznobežné.}$$

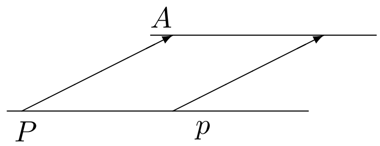
$x = 3 + t_1 = 3 - 2t_2 \implies t_1 = -2t_2$, $y = 2 + 2t_1 = 2 - 4t_2 = -4 + 2t_2 \implies t_2 = 1$, $t_1 = -2$,
$p \cap q = [1,-2,1]$

alebo

$$\begin{aligned} 3 + t_1 &= 3 - 2t_2 & t_1 + 2t_2 &= 0 \\ 2 + 2t_1 &= -4 + 2t_2 & 2t_1 - 2t_2 &= -6 \\ 5 + 2t_1 &= t_2 & 2t_1 - t_2 &= -5 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & -1 & -3 \\ 2 & -1 & -5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & -3 \\ 0 & -5 & -5 \end{array}\right) \implies \begin{array}{c} t_2 = 1 \\ t_1 = -2 \end{array}$$

$\underline{p \cap q = [1,-2,1]}$

b)



$$\overrightarrow{PA} = A - P = (1,1,1) - (3,2,5) = (-2,-1,-4)$$
$$\mathbf{u} = (\ 1,\ \ 2,\ \ 2) \qquad .$$
$$(A-P)\times\mathbf{u} = (6,0,-3) = 3(2,0,-1)$$

$$\mathrm{dist}(A,p) = \frac{\|(A-P)\times\mathbf{u}\|}{\{\mathbf{u}\|} = \frac{3\sqrt{5}}{3} = \sqrt{5}$$

c) $\mathbf{n} = (1,2,2)\times(-2,2,1) = (-2,-5,6)$
$\rho \equiv -2(x-1) - 5(y+2) + 6(z-1) = 0$
$\rho \equiv -2x - 5y + 6z + 2 - 10 - 6 = \underline{-2x - 5y + 6z - 14 = 0}$

d) $$\mathrm{dist}\,(A,\rho) = \frac{|-2-5+6-14|}{\sqrt{4+25+36}} = \frac{15}{\sqrt{65}}$$