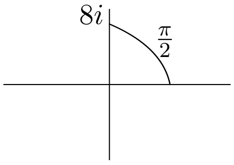
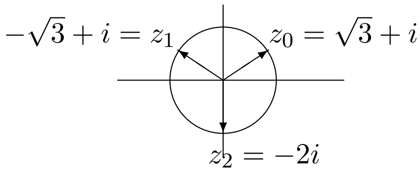
Pr.1) Riešte rovnicu $z^3 = 8i$, riešenie napíšte v algebraickom tvare a znázornite.

Riešenie



$z^3 = 8i = 8e^{i\frac{\pi}{2}} = 8e^{i(\frac{\pi}{2}+k\cdot 2\pi)}, \ k \in \mathbb{Z}$
$z_k = 2e^{\frac{1}{3}(\frac{\pi}{2}+k\cdot 2\pi)} = 2e^{i(\frac{\pi}{6}+k\frac{2\pi}{3})}, \ k = 0, 1, 2$
$z_0 = 2e^{i\frac{\pi}{6}} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \sqrt{3} + i$
$z_1 = 2e^{i\frac{5}{6}\pi} = -\sqrt{3} + i,$
$z_2 = 2e^{i\frac{9}{6}\pi} = -2i$



2) Rozšírenú maticu sústavy upravte na redukovanú stupňovitú a napíšte množinu všetkých jej riešení. Urobte skúšku správnosti.

$$2x_1 - 5x_2 + 3x_3 - x_4 = 1$$
$$3x_1 - 7x_2 + 4x_3 + x_4 = 1$$
$$3x_1 - 4x_2 + x_3 + 2x_4 = 5$$

$$\left(\begin{array}{cccc|c} 2 & -5 & 3 & -1 & 1 \\ 3 & -7 & 4 & 1 & 1 \\ 3 & -4 & 1 & 2 & 5 \end{array}\right) \underset{\substack{r_1 - r_2 \\ r_3 - r_2}}{\sim} \left(\begin{array}{cccc|c} -1 & 2 & -1 & -2 & 0 \\ 3 & -7 & 4 & 1 & 1 \\ 0 & 3 & -3 & 1 & 4 \end{array}\right) \underset{\substack{r_2 + 3r_1 \\ -r_1}}{\sim} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 0 & -1 & 1 & -5 & 1 \\ 0 & 3 & -3 & 1 & 4 \end{array}\right) \underset{\substack{r_3 + 3r_2 \\ -r_2}}{\sim}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 0 & 0 & -14 & 7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right) \underset{\substack{r_1 + 2r_2 \\ r_2 - 5r_3}}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 12 & -2 \\ 0 & 1 & -1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & -1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right)$$

$P = \{(a + 4\,,\, a + 3/2\,,\, a\,,\, -1/2) : a \in \mathbb{R}\}$, skúška:

$L_1 = 2(a+4) - 5(a + \frac{3}{2}) + 3a + \frac{1}{2} = 2a - 5a + 3a + 8 - \frac{15}{2} + \frac{1}{2} = 8 - 7 = 1 = P_1$
$L_2 = 3(a+4) - 7(a + \frac{3}{2}) + 4a - \frac{1}{2} = 3a - 7a + 4a + 12 - \frac{21}{2} - \frac{1}{2} = 1 = P_2$
$L_3 = 3(a+4) - 4(a + \frac{3}{2}) + a - 2\cdot\frac{1}{2} = 12 - 6 - 1 + 3a - 4a + a = 5 = P + 3$

3. [15 bodov] Dané sú matice

$$3A)\quad A=\begin{pmatrix}1&1&1&-2\\-2&0&-1&1\\0&1&0&0\\1&0&-2&-4\end{pmatrix},\qquad B=\begin{pmatrix}1&1&-2\\-2&-1&1\\1&-2&-4\end{pmatrix}$$

Vypočítajte $\det A$, $\det B$, B^{-1}.

Riešenie:

$$\det B=\begin{vmatrix}1&1&-2\\-2&-1&1\\1&-2&-4\end{vmatrix}=(\text{podľa } R_1)\begin{vmatrix}-1&1\\-2&-4\end{vmatrix}-\begin{vmatrix}-2&1\\1&-4\end{vmatrix}-2\begin{vmatrix}-2&-1\\1&-2\end{vmatrix}=6-7-10=-11$$

Rozvoj podľa R_3: $\underline{\det A}=(-1)^{2+3}\begin{vmatrix}1&1&-2\\-2&-1&1\\1&-2&-4\end{vmatrix}=-1\cdot\det B=(-1)\cdot(-11)=\underline{11}$,

$$B^{-1}=\frac{1}{-11}\begin{pmatrix}\begin{vmatrix}-1&1\\-2&-4\end{vmatrix}&-\begin{vmatrix}1&-2\\-2&-4\end{vmatrix}&\begin{vmatrix}1&-2\\-1&1\end{vmatrix}\\-\begin{vmatrix}-2&1\\1&-4\end{vmatrix}&\begin{vmatrix}1&-2\\1&-4\end{vmatrix}&-\begin{vmatrix}1&-2\\-2&1\end{vmatrix}\\\begin{vmatrix}-2&-1\\1&-2\end{vmatrix}&-\begin{vmatrix}1&1\\1&-2\end{vmatrix}&\begin{vmatrix}1&1\\-2&-1\end{vmatrix}\end{pmatrix}=\frac{1}{-11}\begin{pmatrix}6&8&-1\\-7&-2&3\\5&3&1\end{pmatrix}$$

Pr4) Polynóm $f(x)=4x^6-10x^5-4x^4+9x^3+12x^2+4x$ napíšte ako súčin ireducibilných polynómov nad $\mathbb{R}$.

Riešenie: $f(x)=x(4x^5-10x^4-4x^3+9x^2+12x+4)$
$c\in\left\{\pm1,\pm2,\pm4,\pm\frac{1}{2},\pm\frac{1}{4}\right\}$
$f(1)=15,\ f(-1)=-1\cdot(-9)=9$

$$\begin{array}{c|cccccc}
 & 4 & -10 & -4 & 9 & 12 & 4\\
2 & & 8 & -4 & -16 & -14 & -4\\
\hline
 & 4 & -2 & -8 & -7 & -2 & |\underline{0}\\
2 & & 8 & 12 & 8 & 2 & \\
\hline
 & 4 & 6 & 4 & 1 & |\underline{0} & \\
-\frac{1}{2} & & -2 & -4 & -1 & & \\
\hline
 & 4 & 4 & 2 & |\underline{0} & &
\end{array}$$

$f(x)=x(x-2)^2(4x^3+6x^2+4x+1)=(x-2)^2(x+\frac{1}{2})(x^4+4x^2+2)\qquad D=16-32=-16<0$

Pr5

Funkciu $f(x) = \dfrac{4x^4 + 4x^2 - x + 1}{2x^3 - x^2 - 1}$

napíšte ako súčet polynómu a elementárnych zlomkov nad $\mathbb{R}$

$$
\begin{array}{l}
(4x^4 + 4x^2 - x + 1) : (2x^3 - x^2 - 1) = 2x + 1 \\
-\underline{(4x^4 - 2x^3 - 2x)} \\
\qquad 2x^3 + 4x^2 + x + 1 \\
\quad - \underline{(2x^3 - x^2 - 1)} \\
\qquad 5x^2 + x + 2 \qquad\qquad f(x) = 2x + 1 + \dfrac{5x^2 + x + 2}{2x^3 - x^2 - 1}
\end{array}
$$

$$
\begin{array}{c|cccc}
 & 2 & -1 & 0 & -1 \\
1 & & 2 & 1 & 1 \\
\hline
 & 2 & 1 & 1 & |0
\end{array}
\implies 2x^3 - x^2 - 1 = (x-1)(2x^2 + x + 1)
$$

$D = 1 - 8 < 0$

$$\frac{5x^2 + x + 2}{2x^3 - x^2 - 1} = \frac{a}{x-1} + \frac{bx + c}{2x^2 + x + 1}$$

$$
\begin{aligned}
5x^2 + x + 2 = a(2x^2 + x + 1) + (bx + c)(x - 1) &= 2ax^2 + ax + a + bx^2 - bx + cx - c \\
&= (2a + b)x^2 + (a - b + c)x + (a - c)
\end{aligned}
$$

$$
\begin{aligned}
2a + b &= 5 \\
a - b + c &= 1 \\
a - c &= 2
\end{aligned}
$$

$$
\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 1 & -1 & 1 & 1 \\ 2 & 1 & 0 & 5 \end{array}\right) \sim
\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right) \sim
\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 4 & 0 \end{array}\right)
\qquad
\begin{aligned} a &= 2 \\ b &= 1 \\ c &= 0 \end{aligned}
$$

$$\underline{f(x) = 2x + 1 + \frac{2}{x-1} + \frac{x}{2x^2 + x + 1}}$$

Pr.6 Nájdite parametrické rovnice priamky p, ktorá je priesečnicou rovín
$\rho_1 \equiv -4x + y + z + 5 = 0, \qquad \rho_2 \equiv -2x + 2y - z + 7 = 0$
a vypočítajte vzdialenosť bodu $A = (1, 1, 1)$ od priamky p.

$$
\left(\begin{array}{ccc|c} -4 & 1 & 1 & -5 \\ -2 & 2 & -1 & -7 \end{array}\right) \sim
\left(\begin{array}{ccc|c} -6 & 3 & 0 & -12 \\ -2 & 2 & -1 & -7 \end{array}\right) \sim
\left(\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ -2 & 2 & -1 & -7 \end{array}\right) \sim
\left(\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ 0 & 1 & -1 & -3 \end{array}\right)
$$

$$
\begin{aligned}
p \equiv x &= t, \\
y &= -4 + 2t, \quad z = 3 + y \\
z &= -1 + 2t, \quad t \in \mathbb{R}.
\end{aligned}
$$

$P = (0, -4, -1)$, $P - A = (-1, -5, -2)$, $\mathbf{u}_p = (1, 2, 2)$

$$\begin{matrix} -1 & -5 & -2 \\ 1 & 2 & 2 \end{matrix}, \ (P - A) \times u_p = (-6, 0, 3) = 3(-2, 0, 1)$$

$$\operatorname{dist}(A, p) = \frac{\|(P - A) \times u_p\|}{\|\mathbf{u}_p\|} = \tfrac{3\sqrt{5}}{3} = \sqrt{5}$$