SHARP AND MEAGER ELEMENTS IN ORTHOCOMPLETE HOMOGENEOUS EFFECT ALGEBRAS

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ABSTRACT. We prove that every orthocomplete homogeneous effect algebra is sharply dominating. Let us denote the greatest sharp element below x by $x^\downarrow$. For every element x of an orthocomplete homogeneous effect algebra and for every block B with $x \in B$, the interval $[x^\downarrow, x]$ is a subset of B. For every *meager element* (that means, an element x with $x^\downarrow = 0$), the interval $[0, x]$ is a complete MV-effect algebra. As a consequence, the set of all meager elements of an orthocomplete homogeneous effect algebra forms a commutative BCK-algebra with the relative cancellation property. We prove that a complete lattice ordered effect algebra E is completely determined by the complete orthomodular lattice $S(E)$ of sharp elements, the BCK-algebra $M(E)$ of meager elements and a mapping $h : S(E) \to 2^{M(E)}$ given by $h(a) = [0, a] \cap M(E)$.

1. INTRODUCTION

Effect algebras have recently been introduced by Foulis and Bennett in [11] for study of foundations of quantum mechanics. The class of effect algebras includes orthomodular lattices and a subclass equivalent to MV-algebras (see [3]).

In [26], Riečanová proved that every lattice ordered effect algebra is a union of (essentially) MV-algebras. This result is a generalization of the well-known fact that every orthomodular lattice is a union of Boolean algebras. Later, Riečanová and Jenča proved in [21] that the set of all sharp elements of a lattice ordered effect algebra forms an orthomodular lattice. Both papers show that the class of lattice ordered effect algebras generalizes the class of orthomodular lattices in a very natural way. In [17] a new class, called *homogeneous effect algebras* was introduced and most of the results from [26] and [21] were generalized for the homogeneous case. The main result of [17] is that every homogeneous effect algebra is a union of effect algebras satisfying the Riesz decomposition property.

Intuitively, one can consider the class of lattice ordered effect algebras as an "unsharp" generalization of the class of orthomodular lattices and the class of homogeneous effect algebras as an "unsharp" generalization of the class of orthoalgebras (see [12]). In these generalizations, the role of Boolean algebras is played by MV-effect algebras and by effect algebras with the Riesz decomposition property. The problems concerning this generalization were examined, for example, in [27] and [18]. The present paper continues this line of research.

An element x of a lattice ordered effect algebra is sharp if and only if $x \wedge x' = 0$. If E is a complete lattice ordered effect algebra, then the set of all sharp elements

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$S(E)$ forms a complete sublattice of E, closed under arbitrary joins and meets. $S(E)$ is a complete orthomodular lattice. Moreover, it is easy to check that every $x \in E$ allows for a unique decomposition $x = x_S \oplus x_M$, where $x_S \in S(E)$ and 0 is the only sharp element under x_M. Of course, this situation reminds one of the well-known triple representation of Stone algebras, described by C.C. Chen and G. Grätzer in their two-part paper [4], [5]. The main result of this paper is a proof of a similar triple representation theorem for complete lattice ordered effect algebras.

2. Definition and basic relationships

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions.

- (E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
- (E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (E3) If $a \oplus b = a \oplus c$, then $b = c$.
- (E4) If $a \oplus b = 0$, then $a = 0$.
- (E5) For every $a \in E$ there is an $a' \in E$ such that $a \oplus a' = 1$.

Effect algebras were introduced by Foulis and Bennett in their paper [11]. In the original paper, a different but equivalent set of axioms was used.

In their paper [22], Chovanec and Kôpka introduced an essentially equivalent structure called *D-poset*. Another equivalent structure was introduced by Giuntini and Greuling in [13]. We refer to [10] for more information on effect algebras and related topics.

A partial algebra $(E; \oplus, 0)$ satisfying the axioms (E1)-(E4) is called a *generalized effect algebra*.

One can construct examples of effect algebras from an arbitrary partially ordered abelian group $(G, \leq)$ in the following way: Choose any positive $u \in G$; then, for $0 \leq a, b \leq u$, define $a \oplus b$ if and only if $a + b \leq u$ and put $a \oplus b = a + b$. With such partial operation $\oplus$, the interval $[0, u]$ becomes an effect algebra $([0, u], \oplus, 0, u)$. Effect algebras which arise from partially ordered abelian groups in this way are called *interval effect algebras*, see [1].

In a generalized effect algebra E, we write $a \leq b$ if and only if there is $c \in E$ such that $a \oplus c = b$. It is easy to check that for every effect algebra $\leq$ is a partial order on E. Moreover, it is possible to introduce a new partial operation $\ominus$; $b \ominus a$ is defined if and only if $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that, in an effect algebra, $a \oplus b$ is defined if and only if $a \leq b'$ if and only if $b \leq a'$. Therefore, it is usual to denote the domain of $\oplus$ by $\perp$. If $a \perp b$, we say that a and b are *orthogonal*. We say that an element a is *isotropic* if and only if $a \oplus a$ exists. We write shortly

$$n \cdot a := \overbrace{a \oplus \cdots \oplus a}^{n \text{ times}}.$$

The number $\iota(a) = \max\{n \cdot a \text{ exists}\}$ is called *the isotropic index* of a. An isotropic index of a nonzero element need not exist, since it may happen that $n \cdot a$ exists for each $n \in \mathbb{N}$. For such a, we write $\iota(a) = \infty$. If for each nonzero $a \in E$ we have $\iota(a) < \infty$, then we say that E is *archimedean*.

Let E be an effect algebra. Let $E_0 \subseteq E$ be such that $1 \in E_0$ and, for all $a, b \in E_0$ with $a \geq b$, $a \ominus b \in E_0$. Since $a' = 1 \ominus a$ and $a \oplus b = (a' \ominus b)'$, E_0 is closed with respect to $\oplus$ and $'$. We then say that $(E_0, \oplus, 0, 1)$ is a *sub-effect algebra of E*.

Another possibility to construct a substructure of an effect algebra E is to restrict $\oplus$ to a closed interval $[0,a]$, where $a \in E$, letting a act as the unit element. We denote such effect algebra by $[0,a]_E$. Similarly, if P is a generalized effect algebra and Q is a subset of P with the property $x \in Q \Rightarrow [0,x] \subseteq Q$, then the restriction of $\oplus$ to Q is again a generalized effect algebra.

An ideal of a generalized effect algebra P is a subset I of P satisfying the condition

$$a, b \in I \text{ and } a \perp b \Longleftrightarrow a \oplus b \in I.$$

The set of all ideals of a generalized effect algebra P is denoted by $I(P)$. $I(P)$ is a complete lattice with respect to inclusion.

An element c of an effect algebra is *central* (see [14]) if and only if $[0,c]$ is an ideal and, for every $x \in E$, there is a decomposition $x = x_1 \oplus x_2$ such that $x_1 \leq c$, $x_2 \leq c'$. It can be shown that this decomposition is unique. The set $C(E)$ of all central elements of an effect algebra is called *the centre of* E. $C(E)$ is a Boolean algebra. For every central element c of E, E is isomorphic to $[0,c]_E \times [0,c']_E$.

A D-poset is a system $(P; \leq, \ominus, 0, 1)$ consisting of a partially ordered set P bounded by 0 and 1 with a partial binary operation $\ominus$ satisfying the following conditions.

(D1) $b \ominus a$ is defined if and only if $a \leq b$.
(D2) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.
(D3) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

There is a natural, one-to-one correspondence between D-posets and effect algebras. Every effect algebra satisfies the conditions (D1)-(D3). When given a D-poset $(P; \leq, \ominus, 0, 1)$, one can construct an effect algebra $(P; \oplus, 0, 1)$: the domain of $\oplus$ is given by the rule $a \perp b$ if and only if $a \leq 1 \ominus b$ and we then have $a \oplus b = 1 \ominus ((1 \ominus a) \ominus b$. The resulting structure is then an effect algebra with the same $\ominus$ as the original D-poset.

Let E_1, E_2 be effect algebras. A map $\phi : E_1 \mapsto E_2$ is called a *homomorphism of effect algebras* if and only if it satisfies the following condition.

(HE1) $\phi(1) = 1$ and if $a \perp b$, then $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

A homomorphism $\phi : E_1 \mapsto E_2$ of effect algebras is called *full* if and only if the following condition is satisfied.

(HE2) If $\phi(a) \perp \phi(b)$ $\phi(a) \oplus \phi(b) \in \phi(E_1)$ then there exist $a_1, b_1 \in E_1$ such that $a_1 \perp b_1$, $\phi(a) = \phi(a_1)$ and $\phi(b) = \phi(b_1)$.

A bijective, full homomorphism is called an *isomorphism of effect algebras.*

Let D_1, D_2 be D-posets. We say that a mapping $\phi : D_1 \to D_2$ is a *homomorphism of D-posets* if and only if it satisfies the following condition.

(HD1) $\phi(1) = 1$ and if $a \leq b$, than $\phi(a) \leq \phi(b)$ and $\phi(b \ominus a) = \phi(b) \ominus \phi(a)$.

A homomorphism of ϕ is an *isomorphism of D-posets* if and only if ϕ is surjective and $\phi(a) \leq \phi(b)$ implies $a \leq b$.

It is easy to check that ϕ is a homomorphism (isomorphism) of effect algebras if and only if ϕ is a homomorphism (isomorphism) of corresponding D-posets.

Let us note that, for every closed subinterval $[a,b]$ of an effect algebra, the mapping $x \mapsto b \ominus (x \ominus a)$ is an antitone bijection. Thus, every closed subinterval of an effect algebra is a self-dual poset.

An element x is *sharp* if and only if $x \wedge x' = 0$. The set of all sharp elements of an effect algebra E is denoted by $S(E)$. An effect algebra E is *sharply dominating* if and only if, for every element x,

$$x^{\uparrow} := \bigwedge\{t : t \in [x, 1] \cap S(E)\}$$

exists and is sharp. It is easy to see that in a sharply dominating effect algebra E, the element

$$x^{\downarrow} := \bigvee\{t : t \in [0, x] \cap S(E)\}$$

exists and is sharp, for all x. Moreover, we have $(x^{\uparrow})' = (x')^{\downarrow}$ and $(x^{\downarrow})' = (x')^{\uparrow}$. We say that $x^{\uparrow}$ is the *sharp cover of* x and that $x^{\downarrow}$ is the *sharp kernel of* x. In his paper [2], Cattaneo proved that for every sharply dominating effect algebra the set of all sharp elements forms a sub-effect algebra, which is an orthoalgebra. See [15] for another version of the proof.

If E is an effect algebra such that $(E, \leq)$ is a lattice, we say that E is *lattice ordered.*

A finite family of elements $\mathbf{a} = (a_1, \dots, a_n)$ of an effect algebra is called *orthogonal* if and only if $\oplus\mathbf{a} = a_1 \oplus \dots \oplus a_n$ is defined. An infinite family $\mathbf{a} = (a_i)_{i\in S}$ is called *orthogonal* if and only if all finite subfamilies of A are orthogonal. An orthogonal family $\mathbf{a} = (a_i)_{i\in S}$ is called *summable* if and only if

$$\bigoplus \mathbf{a} = \bigvee\{a_{i_1} \oplus \dots \oplus a_{i_n} : \{i_1, \dots, i_n\} \subseteq S\}$$

exists. An effect algebra E is called *κ-orthocomplete* if and only if every orthogonal family of cardinality κ is summable. Every $\aleph_0$-orthocomplete effect algebra is archimedean. The following result was proved in [20] and [19].

Theorem 1. *An effect algebra is κ-orthocomplete if and only if for every chain C with $\operatorname{card}(C) = \kappa$, $\bigvee C$ exists.*

Note that Theorem 1 implies that a lattice ordered effect algebra E is orthocomplete if and only if E is a complete lattice.

Let E be a κ-complete effect algebra, let $A \subseteq E$. We write $\sigma^{\kappa}(A)$ for the set of all $\bigoplus_{i\in S}(a_i)$, where $\operatorname{card}(S) \leq \kappa$ and $(a_i)_{i\in S}$ is an orthogonal family of elements of A.

A finite subset M_F of an effect algebra E is called *compatible with cover in* $X \subseteq E$ if and only if there is a finite orthogonal family $\mathbf{c} = (c_1, \dots, c_n)$ with $Ran(\mathbf{c}) \subseteq X$ such that for every $a \in M_F$ there is a set $A \subseteq \{1, \dots, n\}$ with $a = \bigoplus_{i\in A} c_i$. $\mathbf{c}$ is then called an *orthogonal cover* of M_F. A subset M of E is called *compatible with covers in* $X \subseteq E$ if and only if every finite subset of M is compatible with cover in X. A subset M of E is called *internally compatible* if and only if M is compatible with covers in M. A subset M of E is called *compatible* if and only if M is compatible with covers in E. If $\{a, b\}$ is a compatible set, we write $a \leftrightarrow b$. It is easy to check that $a \leftrightarrow b$ if and only if there are $a_1, b_1, c \in E$ such that $a_1 \oplus c = a$, $b_1 \oplus c = b$, and $a_1 \oplus b_1 \oplus c$ exists. We note that if $a \leq b$ or $a \perp b$ then $a \leftrightarrow b$. In a lattice ordered effect algebra, $a \leftrightarrow b$ if and only if $a \ominus (a \wedge b) \leq b'$ if and only if $a \ominus (a \wedge b) \leq b = (a \vee b) \ominus b$.

A subset M of E is called *mutually compatible* if and only if, for all $a, b \in M$, $a \leftrightarrow b$. Obviously, every compatible subset of an effect algebra is mutually compatible. In the class of lattice ordered effect algebras, the converse also holds. It is well known that a mutually compatible set need not to be compatible (see for example [25]).

An effect algebra satisfying $a \perp a \Longrightarrow a = 0$ is called an *orthoalgebra* (see [12]). An orthoalgebra is an *orthomodular lattice* if and only if it is lattice ordered. An *MV-effect algebra* is a lattice-ordered effect algebra such that for all elements a, b we have $a \leftrightarrow b$. Chovanec and Kôpka proved in [7] that there is a natural, one-to-one correspondence between MV-algebras (introduced by Chang in [3]) and MV-effect algebras. Every MV-effect algebra is an interval in a lattice ordered abelian group (see [23]). We say that an effect algebra satisfies the *Riesz decomposition property* if and only if, for all $u, v_1, \dots, v_n \in E$ such that $v_1 \oplus \dots \oplus v_n$ exists and $u \leq v_1 \oplus \dots \oplus v_n$, there are $u_1, \dots, u_n \in E$ such that,for all $1 \leq i \leq n$, $u_i \leq v_i$ and $u = u_1 \oplus \dots \oplus u_n$. It is easy to check that an effect algebra E has the Riesz decomposition property if and only if E has the Riesz decomposition property with fixed $n = 2$. A lattice ordered effect algebra E satisfies Riesz decomposition property if and only if E is an MV-algebra. An orthoalgebra E satisfies the Riesz decomposition property if and only if E is a Boolean algebra.

An effect algebra E is called *homogeneous* if and only if, for all $u, v_1, \dots, v_n \in E$ such that $u \leq v_1 \oplus \cdots \oplus v_n \leq u'$, there are $u_1, \dots, u_n$ such that, for all $1 \leq i \leq n$, $u_i \leq v_i$ and $u = u_1 \oplus \cdots \oplus u_n$. Similarly as for the Riesz decomposition property, an effect algebra is homogeneous if and only if it satisfies the homogeneity axiom with $n = 2$.

Let E be a homogeneous effect algebra. A subeffect B of E is called *a block* if and only if B is the maximal subeffect algebra of E with the Riesz decomposition property.

The following proposition summarizes some of the results from [17].

Proposition 2.

(a) *Every orthoalgebra is homogeneous.*
(b) *Every lattice ordered effect algebra is homogeneous.*
(c) *An effect algebra E has the Riesz decomposition property if and only if E is homogeneous and compatible.*

Let E be a homogeneous effect algebra.

(d) *A subset B of E is a maximal sub-effect algebra of E with the Riesz decomposition property (such B is called* a block of E*) if and only if B is a maximal internally compatible subset of E.*
(e) *Every finite compatible subset of E is a subset of some block. This implies that every homogeneous effect algebra is a union of its blocks.*
(f) *$S(E)$ is a sub-effect algebra of E.*
(g) *For every block B, $C(B) = S(E) \cap B$.*
(h) *Let $x \in B$, where B is a block of E. Then $\{y : y \leq x, x'\} \subseteq B$.*

In the case of a lattice ordered effect algebra, the blocks are MV-effect algebras, which are sublattices of E (see the main result of [26]). Every mutually compatible subset of a lattice ordered effect algebra can be embedded into a block, hence the blocks are exactly the maximal mutually compatible subsets. In particular, this implies that if $A = \{a, b, c\}$ is a mutually compatible subset, then the sublattice L_A generated by A is mutually compatible, and (since L_A is a sublattice of some block containing A) L_A is a finite distributive lattice. Similarly, if we assume $b \perp c$, then $a \leftrightarrow b \oplus c$.

For homogeneous effect algebras, the situation is a bit more complicated, since we have to deal with internal compatibility if we want to prove that some set of elements is a subset of a block. However, Proposition 2 (e) shows that a finite set of elemets is a subset of a block if and only if it is a compatible set.

As an example of application of the notion of internal compatibility, let us prove the following Theorem.

Theorem 3. *Every chain in a homogeneous effect algebra is a subset of a block.*

Proof. Let E be a homogeneous effect algebra, let $C \subseteq E$ be a chain. Without loss of generality, suppose that $0 \in C$. Let $A = \{x \ominus y : y \leq x \text{ and } x, y \in C\}$. Since $0 \in C$, $C \subseteq A$. We claim that A is internally compatible. Indeed, let A_F be a finite subset of A. There exists a finite chain $C_F \subseteq C$ such that $A_F \subseteq \{x \ominus y : y \leq x \text{ and } x, y \in C_F\}$. Write $C_F = \{c_1, \dots, c_n\}$, where $c_i \leq c_{i+1}$. Then the orthogonal word $\mathbf{c} = (c_{i+1} \ominus c_i : 1 \leq i < n)$ is an orthogonal cover of A_F, with $Ran(\mathbf{c}) \subseteq A$. Thus, A is internally compatible and, by Proposition 2 (d), A is a subset of a block. □

For a lattice ordered effect algebra, $S(E)$ is a sublattice of E (see [21]) and hence an orthomodular lattice. If E is a complete lattice, then for every $X \subseteq S(E)$, $\bigvee X$ and $\bigwedge X$ are sharp. Similarly, every block of E is closed under arbitrary joins and meets. In particular, this implies that every complete lattice ordered effect algebra is sharply dominating (see [21]).

Another type generalization of Riečanová's results from [26] can be found in [8].

3. Sharp elements and infinite sums

The aim of this section is to prove that every orthocomplete homogeneous effect algebra is sharply dominating and examine the behavior of sharp elements with respect to blocks. The main tool we use are certain infinite sums of isotropic elements. Let us introduce some closure operations defined on the set of all subsets of an effect algebra.

Let E be a κ-orthocomplete effect algebra. Let us define a mapping θ^κ on the system of all subsets of E as follows. We write $\theta^\kappa(v)$ for the set of all elements of E of the form $\bigoplus_{i\in S}(u_i)$ or $v \ominus \bigoplus_{i\in S}(u_i)$, where $\operatorname{card}(S) \leq \kappa$ and $(u_i)_{i\in S}$ is an orthogonal family satisfying $\bigoplus(u_i)_{i\in S} \leq v$ and, for all $i \in S$, $v \leq u_i'$. For $A \subseteq E$, we write $\theta^\kappa(A) = \bigcup_{v\in A} \theta^\kappa(v)$. For any set A, $\sigma_I^\kappa(A)$ is the smallest superset of A closed with respect to θ^κ. It is easy to check that $\sigma_I^\kappa(A) = \bigcup_{n=0}^\infty A_i$, where A_i are subsets of E given by the rules $A_0 = A$, $A_{n+1} = \theta^\kappa(A_n)$.

For an orthocomplete effect algebra E and $A \subseteq E$, the symbols $\sigma(A)$ and $\sigma_I(A)$ denote the union of all $\sigma^\kappa(A)$ and $\sigma_I^\kappa(A)$, respectively, where $\kappa \leq \operatorname{card}(E)$.

Proposition 4. *Let E be an κ-orthocomplete homogeneous effect algebra. Let $(v_i)_{i\in S}$ be an orthogonal family with $\operatorname{card}(S) = \kappa$, $u \in E$ be such that*

$$u \leq \bigoplus_{i\in S}(v_i) \leq u'.$$

Then there is an orthogonal family $(u_i)_{i\in S}$ such that $u = \bigoplus_{i\in S}(u_i)$ and $u_i \leq v_i$ for all $i \in S$.

Proof. By the well-ordering principle, we may assume that v_i's are indexed by $\{\alpha : \alpha < \delta\}$, where δ is an ordinal. Without loss of generality, we may assume that $v_0 = 0$. Let us put $v = \bigoplus(v_\alpha)_{\alpha<\delta}$.

It suffices to prove that there is an orthogonal family $(u_\alpha)_{\alpha<\delta}$ such that, for all $\beta < \delta$,

(i) $u_\beta \leq v_\beta$ and
(ii) $u \ominus \bigoplus_{\alpha\leq\beta}(u_\alpha) \leq v \ominus \bigoplus_{\alpha\leq\beta}(v_\alpha)$.

Indeed, since

$$\bigwedge_{\beta<\delta}(u \ominus \bigoplus_{\alpha\leq\beta}(u_\alpha)) = u \ominus \bigvee_{\beta<\delta}\bigoplus_{\alpha\leq\beta}(u_\alpha) = u \ominus \bigoplus_{\alpha<\delta}(u_\alpha)$$

and similarly for the right-hand side of (2), it follows that

$$u \ominus \bigoplus_{\alpha<\delta}(u_\alpha) \leq v \ominus \bigoplus_{\alpha<\delta}(v_\alpha) = 0,$$

hence $u = \bigoplus_{\alpha<\delta}(u_\alpha)$. Reindexing the u_α's and v_α's again by S, we obtain the desired family $(u_i)_{i\in S}$ and the proof is complete.

Let construct the family $(u_\alpha)_{\alpha<\delta}$. For $\beta = 0$, we may put $u_0 = 0$. Suppose that $\gamma > 0$ and that (i) and (ii) are satisfied for all $\beta < \gamma$. Taking infima of both sides of (ii) through all $\beta < \gamma$, we obtain

$$u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha) \leq v \ominus \bigoplus_{\alpha<\gamma}(v_\alpha). \tag{1}$$

Moreover,

$$\begin{aligned} v \ominus \bigoplus_{\alpha<\gamma}(v_\alpha) &= (\bigoplus_{\gamma\leq\alpha<\delta}(v_\alpha) \oplus \bigoplus_{\alpha<\gamma}(v_\alpha)) \ominus \bigoplus_{\alpha<\gamma}(v_\alpha) = \bigoplus_{\gamma\leq\alpha<\delta}(v_\alpha) = \\ &= v_\gamma \oplus \bigoplus_{\gamma<\alpha<\delta}(v_\alpha) = v_\gamma \oplus (v \ominus \bigoplus_{\alpha\leq\gamma}(v_\alpha)) \end{aligned} \tag{2}$$

and

$$v_\gamma \oplus (v \ominus \bigoplus_{\alpha\leq\gamma}(v_\alpha)) \leq u' \leq (u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha))'. \tag{3}$$

From (1)-(3) we obtain

$$u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha) \leq v_\gamma \oplus (v \ominus \bigoplus_{\alpha\leq\gamma}(v_\alpha)) \leq (u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha))'.$$

Since E is homogeneous, there are $u_\gamma, x \in E$ such that $u_\gamma \oplus x = u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha)$, $u_\gamma \leq v_\gamma$ and $x \leq v \ominus \bigoplus_{\alpha\leq\gamma}(v_\alpha)$. Thus,

$$u \ominus \bigoplus_{\alpha\leq\gamma}(u_\alpha) = (u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha)) \ominus u_\gamma = x \leq v \ominus \bigoplus_{\alpha\leq\gamma}(v_\alpha)$$

and the induction step is complete. □

Corollary 5. *Let E be an κ-orthocomplete homogeneous effect algebra. Then $\sigma^\kappa(S(E)) = S(E)$. In particular, $S(E)$ is κ-orthocomplete.*

Proof. Let $(v_i)_{i\in S}$ be an orthogonal family of sharp elements with $\operatorname{card}(S) = \kappa$. Let $u \leq \bigoplus_{i\in S}(v_i)$, $u \leq (\bigoplus_{i\in S}(v_i))'$. This is equivalent to

$$u \leq \bigoplus_{i\in S}(v_i) \leq u'.$$

By Proposition 4, $u = \bigoplus_{i\in S}(u_i)$, where $u_i \leq v_i$. Since, for all $j \in S$,

$$u_j \leq u \leq (\bigoplus_{i\in S}(v_i))' \leq v_j'$$

and $v_j \wedge v_j' = 0$, $u_j = 0$ and hence $u = 0$. □

Proposition 6. *Let E be an κ-orthocomplete homogeneous effect algebra, let $v_1, v_2 \in E$. Let $(u_i)_{i\in S}$, where $\operatorname{card}(S) = \kappa$, be an orthogonal family such that $\bigoplus_{i\in S}(u_i) \leq v_1 \oplus v_2$ and, for all $i \in S$, $v_1 \oplus v_2 \leq u_i'$. Then there are $x_1 \in \theta^\kappa(v_1), x_2 \in \theta^\kappa(v_2)$ such that $x_1 \leq v_1$, $x_2 \leq v_2$, $v_1 \ominus x_1 \in \theta^\kappa(v_1)$, $v_2 \ominus x_2 \in \theta^\kappa(v_2)$ and $x_1 \oplus x_2 = \bigoplus_{i\in S}(u_i)$.*

Proof. Let δ be an ordinal corresponding to κ. We may assume that the u_i's are indexed by the set $\{\alpha : \alpha < \delta\}$. Without loss of generality, we may assume that $u_0 = 0$.

We shall prove that there are families $(x^1_\alpha)_{\alpha<\delta}$, $(x^2_\alpha)_{\alpha<\delta}$ such that for each $\beta < \delta$,

(i) $u_\beta = x^1_\beta \oplus x^2_\beta$ and
(ii) For $j = 1, 2$, $\bigoplus_{\alpha\leq\beta}(x^j_\alpha) \leq v_j$.

For $\beta = 0$, we may put $x^1_0 = x^2_0 = 0$. Let $\gamma > 0$ and suppose that (i) and (ii) are valid for all $\beta < \gamma$. Taking suprema of (ii) through all $\beta < \gamma$, we obtain for $j = 1, 2$

$$\bigoplus_{\alpha<\gamma}(x^j_\alpha) \leq v_j.$$

Since

$$\begin{aligned}\bigoplus_{\alpha<\delta}(u_\alpha) = \bigoplus_{\alpha<\gamma}(u_\alpha) \oplus \bigoplus_{\gamma\leq\alpha<\delta}(u_\alpha) = \bigoplus_{\alpha<\gamma}(x^1_\alpha \oplus x^2_\alpha) \oplus \bigoplus_{\gamma\leq\alpha<\delta}(u_\alpha) = \\ = \bigoplus_{\alpha<\gamma}(x^1_\alpha) \oplus \bigoplus_{\alpha<\gamma}(x^2_\alpha) \oplus \bigoplus_{\gamma\leq\alpha<\delta}(u_\alpha) \leq v_1 \oplus v_2,\end{aligned}$$

we have

$$u_\gamma \leq (v_1 \ominus \bigoplus_{\alpha<\gamma}(x^1_\alpha)) \oplus (v_2 \ominus \bigoplus_{\alpha<\gamma}(x^2_\alpha)) \leq v_1 \oplus v_2 \leq u_\gamma'.$$

Therefore, there are x^1_γ, x^2_γ such that $u_\gamma = x^1_\gamma \oplus x^2_\gamma$ and for $j = 1, 2$, $x^j_\gamma \leq (v_j \ominus \bigoplus_{\alpha<\gamma}(x^j_\alpha))$. Hence, $\bigoplus_{\alpha\leq\gamma}(x^j_\alpha) \leq v_j$, $j = 1, 2$, and the induction is complete.

Let us put $x_j = \bigoplus_{\alpha<\delta}(x^j_\alpha)$, where $j = 1, 2$. We have $\bigoplus_{\alpha<\delta}(u_\alpha) = x_1 \oplus x_2$ and, since

$$x_j = \bigvee_{\beta<\delta}\bigoplus_{\alpha\leq\beta} x^j_\alpha,$$

we see that (ii) implies that $x_1 \leq v_1$ and $x_2 \leq v_2$.

Since, for each $\alpha < \delta$ and for $j = 1, 2$, $x^j_\alpha \leq u_\alpha \leq (v_1 \oplus v_2)' \leq v_j'$, we have $x_j, v_j \ominus x_j \in \theta^\kappa(v_j)$. □

Corollary 7. *Let E be an κ-orthocomplete homogeneous effect algebra, let $v_1, \dots, v_k \in E$. Let $(u_i)_{i\in S}$, where $\operatorname{card}(S) = \kappa$, be an orthogonal family such that $\bigoplus_{i\in S}(u_i) \leq v_1 \oplus \dots \oplus v_k$ and, for all $i \in S$, $v_1 \oplus \dots \oplus v_k \leq u_i'$. Then there are $x_1, \dots, x_k$ such that $x_j \leq v_j$ and $x_j, v_j \ominus x_j \in \theta^\kappa(v_j)$ for $j = 1, \dots, k$ and $x_1 \oplus \dots \oplus x_k = \bigoplus_{i\in S}(u_i)$.*

Proof. The proof is a straightforward induction with respect to k and is therefore omitted. □

Theorem 8. *Let E be a κ-orthocomplete homogeneous effect algebra. For every internally compatible subset A of E, $\sigma^\kappa_I(A)$ is internally compatible.*

Proof. We have $\sigma_I^\kappa(A) = \bigcup_{i=0}^\infty A_i$, where $A_0 = A$ and $A_{n+1} = \theta^\kappa(A_n)$. Since, for all $n \in \mathbb{N}$, $A_n \subseteq A_{n+1}$, every finite subset of $\sigma_I^\kappa(A)$ can be embedded into some A_n. Thus, it suffices to prove that every A_n is compatible with support in $\sigma_I^\kappa(A)$. By assumption, $A_0 = A$ is internally compatible. Assume that, for some $n \in \mathbb{N}$, A_n is compatible with support in $\sigma_I^\kappa(A)$. Obviously, every finite subset of $A_{n+1} = \theta^\kappa(A_n)$ can be embedded into a subset Y_k of the form

$$Y_k = \{\bigoplus_{i\in S_1}(u_i^1), v_1 \ominus \bigoplus_{i\in S_1}(u_i^1), \dots, \bigoplus_{i\in S_k}(u_i^k), v_k \ominus \bigoplus_{i\in S_k}(u_i^k)\},$$

where $\{v_1,\dots,v_k\} \subseteq A_n$ and, for $j = 1,\dots,k$, $\operatorname{card}(S_j) \leq \kappa$, and $(u_i^j)_{i\in S}$ is an orthogonal family with $v_j \leq u_i'$.

We now prove the following

Claim. Let Y_k be as above. For every cover $\mathbf{c}_0$ of $\{v_1,\dots,v_k\}$, there is a refinement $\mathbf{w}$ of $\mathbf{c}_0$ such that $\mathbf{w}$ covers Y_k and $\operatorname{Ran}(\mathbf{w}) \subseteq \sigma_I^\kappa(\operatorname{Ran}(\mathbf{c}_0))$.

For $k = 0$, we may put $\mathbf{w} = \mathbf{c}_0$. Assume that the Claim is satisfied for some k. Let $\mathbf{c}_0$ be a cover of $\{v_1,\dots,v_{k+1}\} \subseteq A_n$. Since $\mathbf{c}_0$ is the cover $\{v_1,\dots,v_k\}$ as well, by the induction hypothesis there is a refinement of $\mathbf{c}_0$, say $\mathbf{c}_1$ such that $\mathbf{c}_1$ covers Y_k and $\operatorname{Ran}(\mathbf{c}_1) \subseteq \sigma_I^\kappa(\operatorname{Ran}(\mathbf{c}_0))$. As $\mathbf{c}_1$ is a refinement of $\mathbf{c}_0$, $\mathbf{c}_1$ covers $\{v_1,\dots,v_{k+1}\}$. Thus, there are $(c_1,\dots,c_m) \subseteq \mathbf{c}_1$ such that $v_{k+1} = c_1 \oplus \dots \oplus c_m$. By Corollary 7, there are $x_1,\dots,x_m$ such that $\bigoplus_{i\in S_{k+1}}(u_i^{k+1}) = x_1 \oplus \dots \oplus x_m$ and, for all $t = 1,\dots,m$, $x_t \leq c_t$ and $x_t, c_t \ominus x_t \in \theta^\kappa(c_t)$. Let us construct a refinement $\mathbf{w}$ of $\mathbf{c}_1$ by replacing each of the c_t's by the pair $(x_t, c_t \ominus x_t)$. Then $\mathbf{w}$ is a refinement of $\mathbf{c}_0$, $\mathbf{w}$ covers Y_{k+1} and

$$\operatorname{Ran}(\mathbf{w}) \subseteq \theta^\kappa(\operatorname{Ran}(\mathbf{c}_1)) \subseteq \theta^\kappa(\sigma_I^\kappa(\operatorname{Ran}(\mathbf{c}_0))) = \sigma_I^\kappa(\operatorname{Ran}(\mathbf{c}_0)).$$

Let F be a finite subset of A_{n+1}. Then F can be embedded into some $Y_k \subseteq A_{n+1}$. By the outer induction hypothesis, A_n is compatible with covers in $\sigma_I^\kappa(A)$, thus $\{v_1,\dots,v_k\}$ is compatible with cover in $\sigma_I^\kappa(A)$. Let $\mathbf{c}$ be an orthogonal cover of $\{v_1,\dots,v_k\}$ with $\operatorname{Ran}(\mathbf{c}) \subseteq \sigma_I^\kappa(A)$. By the Claim, there is a refinement $\mathbf{w}$ of $\mathbf{c}$, such that $\mathbf{w}$ covers Y_k and $\operatorname{Ran}(\mathbf{w}) \subseteq \sigma_I^\kappa(\operatorname{Ran}(\mathbf{c})) \subseteq \sigma_I^\kappa(A)$. Thus, F is compatible with covers in $\sigma_I^\kappa(A)$ and we see that $\sigma_I^\kappa(A)$ is internally compatible. □

Corollary 9. *Let E be a κ-orthocomplete homogeneous effect algebra. For every block B of E, $\sigma_I^\kappa(B) = B$.*

Proof. By Proposition 2, blocks of E coincide with maximal internally compatible subsets of E containing 1. The rest follows by Theorem 8. □

Corollary 10. *Let E be an orthocomplete homogeneous effect algebra. For every block B of E, $\sigma_I(B) = B$.*

Problem 11. Let E be an κ-orthocomplete homogeneous effect algebra, let A be an internally compatible subset of E. Is it true that $\sigma^\kappa(A)$ is internally compatible? Equivalently, is it true that the blocks of E are closed with respect to σ^κ?

Proposition 12. *Let E be a κ-orthocomplete homogeneous effect algebra, let $x \in E$ and let B be a block of E with $x \in B$. Let $(x_i)_{i\in S}$ be an orthogonal family with $\operatorname{card}(S) = \kappa$ such that, for all $i \in S$, $x_i \leq x'$ and $\bigoplus_{i\in S}(x_i) \leq x$. For every $u \leq \bigoplus_{i\in S}(x_i) \leq x$, there exists a family $(u_i)_{i\in S}$ such that $u = \bigoplus_{i\in S}(u_i) \in B$ and for all $i \in S$, $u_i \leq x_i$.*

Proof. We may suppose that the x_i's are indexed by $\{\alpha : \alpha < \delta\}$, where δ is some ordinal and that $x_0 = 0$. It suffices to prove prove that there exists a family $(u_\alpha)_{\alpha<\delta}$ such that, for each $\beta < \delta$,

(i) $u_\alpha \leq x_\alpha$
(ii) $u \ominus \bigoplus_{\alpha\leq\beta}(u_\alpha) \leq \bigoplus_{\alpha>\beta}(x_\alpha)$.

This implies that $u = \bigoplus_{\alpha<\delta}(u_\alpha) \in \sigma_I^\kappa(x)$, and we obtain $u \in B$ by Corollary 9.

For $\beta = 0$, we may put $u_0 = 0$. Suppose that $\gamma > 0$ and that (i) and (ii) are satisfied for all $\beta < \gamma$. Similarly as in the proof of Proposition 4, we obtain

$$u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha) \leq \bigoplus_{\alpha\geq\gamma}(x_\alpha)$$

. Since $u \leq x$, there exists a block B with $x, u \in B$. By Corollary 9, $\bigoplus_{\alpha<\gamma}(u_\alpha), \bigoplus_{\alpha\geq\gamma}(x_\alpha) \in B$. Since B satisfies the Riesz decomposition property, there exist u_α, z such that $u \ominus \bigoplus_{\alpha<\gamma}(u_\alpha) = u_\alpha \oplus z$, $u_\alpha \leq x_\alpha$ and $z \leq \bigoplus_{\alpha>\gamma}(x_\alpha)$. It remains to observe that $z = u \ominus \bigoplus_{\alpha\leq\gamma}(u_\alpha)$. □

Theorem 13. *Let E be an orthocomplete homogeneous effect algebra, let $x \in E$. Let $(x_i)_{i\in S}$ be a maximal orthogonal family such that, for all $i \in S$, $x_i \leq x'$ and $\bigoplus_{i\in S}(x_i) \leq x$. Then $x^\downarrow = x \ominus \bigoplus_{i\in S}(x_i)$.*

Proof. Let us prove $x \ominus \bigoplus_{i\in S}(x_i) \in E_S$ first. Let $r \in E$ be such that

$$r \leq x \ominus \bigoplus_{i\in S}(x_i), (x \ominus \bigoplus_{i\in S}(x_i))',$$

that means,

$$r \leq x \ominus \bigoplus_{i\in S}(x_i) \leq r'.$$

This is equivalent to

$$r \leq (x \ominus \bigoplus_{i\in S}(x_i))' = x' \oplus \bigoplus_{i\in S}(x_i) \leq r',$$

hence there are $r_1, r_2 \in E$ such that $r_1 \leq x'$, $r_2 \leq \bigoplus_{i\in S}(x_i)$, $r = r_1 \oplus r_2$. Since

$$r_1 \leq r \leq x \ominus \bigoplus_{i\in S}(x_i)$$

and

$$r_1 \oplus \bigoplus_{i\in S}(x_i) \leq x,$$

we have $r_1 = 0$ by maximality of $(x_i)_{i\in S}$. Since

$$r_2 \leq \bigoplus_{i\in S}(x_i) \leq r' \leq r_2',$$

Theorem 4 implies that there is an orthogonal family $(u_i)_{i\in S}$ such that $r_2 = \bigoplus_{i\in S}(u_i)$ and $u_i \leq x_i$ for all $i \in S$. For every $i \in S$ we have

$$u_i \leq r_2 \leq r \leq x \ominus \bigoplus_{i\in S}(x_i),$$

hence

$$u_i \oplus \bigoplus_{i\in S}(x_i) \leq x.$$

Since $u_i \leq x_i \leq x'$, this implies that, for all $i \in S$, $u_i = 0$ and $r_2 = 0$. Thus, $r = r_1 \oplus r_2 = 0$ and $x \ominus \bigoplus_{i\in S}(x_i) \in E_S$.

Let $t \in E_S \cap [0, x]$. We shall prove that $t \leq x \ominus \bigoplus_{i\in S}(x_i)$. Again, we may suppose that the x_i's are indexed by $\{\alpha : \alpha < \delta\}$, where δ is some ordinal. We shall prove that, for all $\beta < \delta$,

$$t \leq x \ominus \bigoplus_{\alpha\leq\beta}(x_\alpha).$$

For $\beta = 0$, there is nothing to prove. Let $\gamma < \delta$ and suppose the induction hypothesis for all $\beta < \gamma$. This implies that

$$t \leq x \ominus \bigoplus_{\alpha<\gamma}(x_\alpha) = (x \ominus \bigoplus_{\alpha\leq\gamma}(x_\alpha)) \oplus x_\gamma.$$

Since $\{x, t\}$ is a finite compatible set, there exists a block B with $x, t \in B$. By Theorem 8, $x \ominus \bigoplus_{\alpha\leq\gamma}(x_\alpha), x_\gamma \in B$. Therefore, $t = t_1 \oplus t_2$, where $t_1 \leq x \ominus \bigoplus_{\alpha\leq\gamma}(x_\alpha)$, $t_2 \leq x_\gamma$. However, since $t_2 \leq x_\gamma \leq x' \leq t'$ and $t \in E_S$, we see that $t_2 = 0$. This completes the induction step.

It remains to observe that,

$$t \leq x \ominus \bigoplus_{\alpha<\delta}(x_\alpha) = x \ominus (x_i)_{i\in S}.$$

□

Corollary 14. *Every orthocomplete homogeneous effect algebra is sharply dominating. Moreover, for every block B, $x \in B$ implies that $[x^\downarrow, x], [x, x^\uparrow] \subseteq B$.*

Proof. By Theorem 13 and Corollary 9, $x^\downarrow$ exists and $x^\downarrow \in B$. For $x^\uparrow \in B$, it suffices to observe that $x' \in B$ and that $x^\uparrow = ((x')^\downarrow)'$. Let $y \in [x^\downarrow, x]$. Let $(x_i)_{i\in S}$ be as in Theorem 13. We have $y \ominus x^\downarrow \leq \bigoplus_{i\in S}(x_i)$. By Proposition 12, $y \ominus x^\downarrow \in B$ and hence $y = x^\downarrow \oplus (y \ominus x^\downarrow) \in B$.

Let $y \in [x, x^\uparrow]$. This implies that $y' \leq x'$ and $y' \geq (x^\uparrow)' = (x')^\downarrow$. Since $x \in B$, $x' \in B$. By above part of the proof $y' \in [(x')^\downarrow, x']$ and $x' \in B$ imply that $y' \in B$. Since $y' \in B$, $y \in B$. □

4. Meager and dense elements

Borrowing the terminology from the theory of Stone algebras, we say that an element x of a homogeneous effect algebra is *dense* if and only if $x^\uparrow = 1$. The set of all dense elements of an effect algebra E is denoted by $D(E)$. An element x of an effect algebra is *meager* if and only if $x^\downarrow = 0$. The set of all meager elements is denoted by $M(E)$. It is easy to check that

$$D(E) = \{x' : x \in M(E)\}.$$

We note that, for $a \in S(E)$, $M([0, a]_E) = M(E) \cap [0, a]$ and $D([0, a]_E) = \{x : x^\uparrow = a\}$.

Proposition 15. *Let E be a sharply dominating effect algebra. Then every $x \in E$ has a unique decomposition $x = x_S \oplus x_M$, where $x_S \in S(E)$ and $x_M \in M(E)$.*

Proof. The element $x \ominus x^\downarrow$ is meager. Indeed, suppose that there is a sharp element a with $a \leq x \ominus x^\downarrow$. By [2], $S(E)$ is a subeffect algebra of E. Therefore $x^\downarrow \oplus a \in S$. However, since $x^\downarrow \oplus a \leq x$, $x^\downarrow \oplus a \leq x^\downarrow$. Thus $a = 0$.

Suppose that there is a decomposition $x = x_S \oplus x_M$ with $x_S \in S(E)$ and $x_M =\in M(E)$. We have

$$x^{\downarrow} \oplus (x \ominus x^{\downarrow}) = x = x_S \oplus x_M.$$

Since $x_S \leq x^{\downarrow}$, $x_M = (x^{\downarrow} \ominus x_S) \oplus (x \ominus x^{\downarrow})$. Since x_M is meager and $x^{\downarrow} \ominus x_S \in S(E)$, $x^{\downarrow} \ominus x_S = 0$. Thus, $x^{\downarrow} = x_S$ and $x \ominus x^{\downarrow} = x_M$. □

Proposition 16. *Let E be an orthocomplete homogeneous effect algebra, let $x \in M(E)$. Then $[0, x]_E$ is a complete MV-effect algebra.*

Proof. Obviously, $[0, x]_E$ is orthocomplete. Since (see [19]) every orthocomplete effect algebra satisfying the Riesz decomposition property is a complete MV-effect algebra, it suffices to prove that $[0, x]_E$ satisfies the Riesz decomposition property. Let B be a block with $x \in B$. By Corollary 14, $[0, x] \subseteq B$. Since B satisfies the Riesz decomposition property, $[0, x]_E$ satisfies the Riesz decomposition property. □

Proposition 17. *For every orthocomplete homogeneous effect algebra, $M(E)$ is a meet semilattice.*

Proof. Let $x, y \in M(E)$. Since E is chain-complete, every lower bound of $\{x, y\}$ is under a maximal lower bound of $\{x, y\}$. Let z_1, z_2 be maximal lower bounds of $\{x, y\}$. By Proposition 16, $[0, x]_E$ is an MV-effect algebra. Therefore, the elements z_1, z_2 have a meet $z_1 \wedge_x z_2$ in $[0, x]$. Similarly, $z_1 \wedge_y z_2$ exists and it is easy to see that $z_1 \wedge_x z_2 = z_1 \wedge_y z_2$. Since both $[0, x]_E$ and $[0, y]_E$ are MV-effect algebras, this implies that

$$z_1 \vee_x z_2 = z_1 \oplus (z_2 \ominus (z_1 \wedge_x z_2)) = z_1 \oplus (z_2 \ominus (z_1 \wedge_y z_2)) = z_1 \vee_y z_2.$$

Since z_1, z_2 are maximal lower bounds of $\{x, y\}$, this implies that $z_1 = z_2$. □

Definition 18. [28] An algebra $(X; *, 0)$ of type $(2, 0)$ is a *commutative BCK-algebra* if and only if the following conditions are satisfied.

(i) $x * (x * y) = y * (y * x)$
(ii) $(x * y) * z = (x * z) * y$
(iii) $x * x = 0$
(iv) $x * 0 = x$

On every commutative BCK-algebra X, we can define a partial order given by the rule $a \leq b$ if and only if $a * b = 0$. In this partial order, X is a lower semilattice with 0.

An important subclass of commutative BCK-algebras is the class of commutative BCK-algebras satisfying the *relative cancellation property*:

(v) For all $a, x, y \in X$, $a \leq x, y$ and $x * a = y * a$ imply $x = y$.

The relative cancellation property was introduced in [9].

It follows from [24] that every MV-effect algebra is a commutative BCK-algebra with the relative cancellation property when we define $x * y = x \ominus (x \wedge y)$. Moreover, every upper-bounded BCK-algebra arises from an MV-effect algebra in this way.

Proposition 19. *Let E be an orthocomplete homogeneous effect algebra. For $x, y \in M(E)$, define $x * y = x \ominus x \wedge y$. Then $(M(E); *, 0)$ is a commutative BCK-algebra with the relative cancellation property.*

Proof.

(i): We have $x*(x*y) = x\ominus x\wedge(x*y) = x\ominus x\wedge(x\ominus x\wedge y) = x\ominus(x\ominus x\wedge y) = x\wedge y$.

(ii): Let us first note, that every interval $[0,x]$ is a $*$-subalgebra of $M(E)$ and, since $[0,x]_E$ is an MV-effect algebra, $([0,x];*,0)$ is a commutative BCK-algebra. Write $y_x = x\wedge y$ and $z_x = z\wedge y$. We have

$$(x*y)*z = (x*y)\ominus(x*y)\wedge z = (x\ominus x\wedge y)\ominus(x\ominus x\wedge y)\wedge z =$$
$$= (x\ominus x\wedge y_x)\ominus(x\ominus x\wedge y_x)\wedge z_x = (x*y_x)*z_x$$

and, similarly, $(x*z)*y = (x*z_x)*y_x$. Since $[0,x]$ is a commutative BCK-algebra, $(x*y_x)*z_x = (x*z_x)*y_x$.

The remaining conditions are clearly satisfied. □

Lemma 20. *Let E be a complete MV-effect algebra, let $u,v\in E$, $u\wedge v = 0$. Then $u^\uparrow\wedge v = 0$.*

Proof. We have $u^\uparrow\ominus u = \bigoplus_{i\in S} u_i$, where $u_i\leq u,u'$. Suppose that $w\leq u^\uparrow, v$. As $w\leq u\oplus(u^\uparrow\ominus u)$, there are $y\leq u$, $z\leq u^\uparrow\ominus u$ such that $w = y\oplus z$. Since $u\wedge v = 0$, $y = 0$. By Proposition 12 (put $x := u'$), $z = \bigoplus_{i\in S}(z_i)$ with $z_i\leq u,u'$. However, for all $i\in S$, $z_i\leq z\leq w\leq v$ and $u\wedge v = 0$, hence $z = 0$ and $w = 0$. □

Proposition 21. *For every complete MV-effect algebra E, $M(E)$ is a sublattice of E.*

Proof. By Proposition 17, $M(E)$ is a meet subsemilattice of E. Let $x,y\in M(E)$. Let $t\in S(E)$, $t\leq x\vee y$. Since E is a distributive lattice, there are u,v such that $u\leq x$, $v\leq y$, $t = u\vee v$. As $t\in S(E)$,

$$0 = t\wedge t' = (u\vee v)\wedge(u\vee v)' = (u\wedge u'\wedge v')\vee(v\wedge v'\wedge u') = 0$$

and we see that $u\wedge u'\wedge v' = 0$. By Lemma 20, $(u\wedge u')\wedge v' = 0$ implies that $(u\wedge u')\wedge(v')^\uparrow = 0$. However, since $v\in M(E)$, $v'\in D(E)$ and $(v')^\uparrow = 1$. Therefore, $u\wedge u' = 0$ and $u = 0$. Similarly, $v = 0$ and $t = u\vee v = 0$. This implies that $(x\vee y)^\downarrow = 0$, that means, $x\vee y\in M(E)$. □

Lemma 22. *Let E be a complete lattice ordered effect algebra, let B be a block of E. Then $M(B)\subseteq M(E)$.*

Proof. Let $x\in M(B)$. Suppose that there is $a\in S(B)\subseteq S(E)$ with $a\leq x$. Then $a\leq x^\downarrow$. But $x^\downarrow\in B$ and $x^\downarrow\in S(B)$, hence $x^\downarrow = 0$ and $a = 0$. □

Proposition 23. *Let E be a complete lattice ordered effect algebra, let $x,y\in M(E)$. Then $x\leftrightarrow y$ if and only if $x\vee y\in M(E)$.*

Proof. Suppose that $x\leftrightarrow y$. Let B be a block of E with $x,y\in B$. Since B is a complete MV-effect algebra, $x\vee y\in M(B)\subseteq M(E)$.

Suppose that $x\vee y\in M(E)$. By Proposition 16, $[0,x\vee y]_E$ is an MV-effect algebra. Therefore, $x\leftrightarrow y$. □

Proposition 24. *Let E be a complete lattice ordered effect algebra, let $x,y\in D(E)$. Then $x\leftrightarrow y$ if and only if $x\wedge y\in D(E)$.*

Proof. We have $x',y'\in M(E)$ and $x\leftrightarrow y$ if and only if $x'\leftrightarrow y'$ if and only if $x'\vee y'\in M(E)$ if and only if $(x'\vee y')' = x\wedge y\in D(E)$. □

5. A triple representation

In this section, E denotes a complete lattice ordered effect algebra. Let us associate with E a triple $(S(E), M(E), h)$, where $h : S(E) \to 2^{M(E)}$ is given by $h(a) = \{x \in M(E) : x \leq a\}$. The aim of this section is to prove the following theorem.

Theorem 25. *The triple $(S(E), M(E), h)$ characterizes E up to isomorphism.*

To prove it, we shall construct a isomorphic copy of the original effect algebra E from the triple. To construct a isomorphic copy of E, we need to construct the following mappings in terms of the triple.

(M1) The mapping $^{\uparrow} : M(E) \to S(E)$.
(M2) For every $a \in S(E)$, a mapping $M(E) \to h(a)$, which is given by $x \mapsto x \wedge a$.
(M3) The mapping $S : M(E) \to M(E)$ given by $S(x) = x^{\uparrow} \ominus x$.

The first mapping (M1) is easy to reconstruct: it is clear that, for all $x \in M(E)$,

$$x^{\uparrow} = \bigwedge\{a \in S(E) : x \in h(a)\}.$$

Let us proceed with the mapping (M2). We recall (see [16]), that a *subcentral ideal* of a generalized effect algebra P is an ideal I such that there exists an ideal I', called *the complement of* I such that for all $x \in P$ there is a unique decomposition $x = x_I \oplus x_{I'}$, where $x_I \in I$, $x_{I'} \in I'$. The set of all subcentral ideals is denoted by $SCI(P)$. As a sublattice of $I(P)$, $SCI(P)$ is a Boolean lattice. It follows from the results of [16] and [6] that the ideal lattice of every generalized effect algebra P satisfying the Riesz decomposition property is distributive and that $SCI(P)$ is the centre of the ideal lattice of P.

The uniqueness of the decomposition $x = x_I \oplus x_{I'}$ allows us to associate with every subcentral ideal I a projection mapping $\pi_I : P \to I$ given by $\pi_I(x) = x_I$. This mapping is a morphism of generalized effect algebras.

Proposition 26. *For every $a \in S(E)$, $h(a)$ is a subcentral ideal of $M(E)$. Moreover, for all $y \in M(E)$, $\pi_{h(a)}(y) = y \wedge a$.*

Proof. By the remarks in the previous paragraph, it suffices to prove that $h(a)$ is a complemented element of $I(M(E))$. Let us prove that $h(a) \in I(M(E))$. Let $x_1, x_2 \in h(a)$, $x_1 \perp_{M(E)} x_2$. Since $\{a, x_1, x_2\}$ is a mutually compatible set, there exists a block $B \supseteq \{a, x_1, x_2\}$ and we have $x_1 \oplus x_2 \in B$. Since $a \in S(E) \cap B$, $a \in C(B)$ and $[0, a] \cap B$ is an ideal of B. Therefore, $x_1 \oplus x_2 \leq a$ and $x_1 \oplus x_2 \in h(a)$.

Let us prove that there is a complement of $h(a)$ in the ideal lattice of $M(E)$. We claim that

$$I := \{x \in M(E) : x \wedge a = 0\}$$

is the complement of $h(a)$. Obviously, $h(a) \cap I = \{0\}$. It remains to prove that I is an ideal of $M(E)$ and that, in the lattice $I(M(E))$, $h(a) \vee I = M(E)$. Let $y_1, y_2 \in I$, $y_1 \perp_{M(E)} y_2$. Suppose that $x \leq y_1 \oplus y_2$, $x \leq a$. Since $[0, y_1 \oplus y_2]_E$ satisfies the Riesz decomposition property, there are $x_1 \leq y_1$, $x_2 \leq y_2$ such that $x = x_1 \oplus x_2$. Since $y_1 \wedge a = y_2 \wedge a = 0$, we see that $x_1 = x_2 = 0$. Thus, $(y_1 \oplus y_2) \wedge a = 0$, $y_1 \oplus y_2 \in I$ and I is an ideal of $M(E)$.

Let $y \in M(E)$. Then $y = (a \wedge y) \oplus [y \ominus (a \wedge y)]$. We claim that $y \ominus (a \wedge y) \in I$. Suppose that $x \leq a, x \leq y \ominus (a \wedge y)$. As $x \perp a \wedge y$, $x \leq a$ and $a \wedge y \leq y$, the set $\{a, x, a \wedge y\}$ is mutually compatible and can be embedded into a block B. We see that $x \oplus (a \wedge y) \leq y$. Since a is sharp, a is central in B and $[0, a]_B$ is an ideal of B. Thus $x \oplus (a \wedge y) \leq a$ and $x \leq y \ominus (a \wedge y), a \ominus (a \wedge y)$. However, $[y \ominus (a \wedge y)] \wedge [a \ominus (a \wedge y)] = 0$, so $x = 0$. We have proved that $[y \ominus (a \wedge y)] \wedge a = 0$, that means, $y \ominus (a \wedge y) \in I$. Thus, every $y \in M(E)$ is a sum of $a \wedge y \in h(a)$ and $y \ominus (a \wedge y) \in I$. This implies that $h(a) \vee I = M(E)$. Note that we have proved $\pi_{h(a)}(y) = y \wedge a$ as well. □

Proposition 26 shows that the mapping (M2) is equal to $\pi_{h(a)}$. Since $\pi_{h(a)}$ is given in terms of the triple, we can construct the mapping (M2) from the triple.

Let us proceed with the mapping (M3). To simplify the proof, let us first deal with the case $x^\uparrow = 1$.

Lemma 27. *Let E be a complete lattice ordered effect algebra, let $x \in M(E) \cap D(E)$. Then $y = x'$ is the only element such that*

(i) $y \in M(E) \cap D(E)$

(ii) $y \leftrightarrow_{M(E)} x$

(iii) *For all $z \in M(E)$, $z \perp_{M(E)} x$ if and only if $z \leq y$ and $y \ominus z \in D(E)$.*

Proof. Let us prove that $y = x'$ satisfies (i)–(iii). Clearly, (i) is satisfied. By Corollary 23, $x \leftrightarrow x'$ implies that $x \vee x' \in M(E)$. By Proposition 16, x, x' are compatible in $[0, x \vee x']_E$ and hence also in $M(E)$. We see that $y = x'$ satisfies (ii). To prove (iii), let $z \in M(E)$ and suppose that $z \perp_{M(E)} x$. As $z \perp x$, $z \leq x'$. As $x \oplus z \in M(E)$, $(x \oplus z)' = x' \ominus z \in D(E)$. The proof of the reverse implication of (iii) is similar.

Let us prove that $x' = y$ is the only element satisfying (i)-(iii). Let y_1, y_2 be such that for $y = y_1, y_2$ (i) and (iii) are satisfied and suppose that $y_1 \leftrightarrow y_2$. Put $t := y_1 \ominus (y_1 \wedge y_2)$. We shall prove the following Claim.

Claim. For all $n \in \mathbb{N}$, $x \perp_{M(E)} n \cdot t$ and $n \cdot t \leq y_1, y_2$.

Proof of the Claim. For $n = 0$, there is nothing to prove. Suppose that the Claim is true for some $n \in \mathbb{N}$. Note that

$$t = y_1 \ominus (y_1 \wedge y_2) = (y_1 \ominus n \cdot t) \ominus [(y_1 \ominus n \cdot t) \wedge (y_2 \ominus n \cdot t)].$$

This implies that $t \leq y_1 \ominus n \cdot t$ and $(n+1) \cdot t \leq y_1$. Moreover,

$$y_1 \ominus (n+1) \cdot t = (y_1 \ominus n \cdot t) \ominus t = (y_1 \ominus n \cdot t) \wedge (y_2 \ominus n \cdot t).$$

As $x \perp_{M(E)} n \cdot t$ and (iii) is satisfied for $y = y_1, y_2$, we see that $y_1 \ominus n \cdot t, y_2 \ominus n \cdot t$ are dense. Since $y_1 \ominus n \cdot t \leftrightarrow y_2 \ominus n \cdot t$, we may apply Corollary 24 to prove that $(y_1 \ominus n \cdot t) \wedge (y_2 \ominus n \cdot t) = y_1 \ominus (n+1) \cdot t \in D(E)$. Therefore, $x \perp_{M(E)} (n+1) \cdot t$.

Since $y = y_2$ satisfies (iii), this implies that $(n+1) \cdot t \leq y_2$ and the Claim is proved.

Since E is orthocomplete, E is archimedean. Thus, the existence of $n \cdot t$ for all $n \in \mathbb{N}$ implies that $t = 0$. We have proved that $y_1 \ominus (y_1 \wedge y_2) = 0$, that means, $y_1 \leq y_2$. We omit the proof of $y_2 \leq y_1$, because it is completely symmetric to the above proof. We see that $y_1 = y_2$

Suppose that some y satisfies (i)–(iii) and put $y_1 := y$, $y_2 = x'$. It follows that $y = x'$. □

Corollary 28. *Let E be a complete MV-effect algebra and let $x \in M(E)$. Then $y = x^\uparrow \ominus x$ is the only element satisfying the conditions*

(i) $y \in h(x^\uparrow)$, $y^\uparrow = x^\uparrow$.

(ii) $y \leftrightarrow_{h(x^\uparrow)} x$

(iii) *For all $z \in h(x^\uparrow)$, $z \perp_{h(x^\uparrow)} x$ if and only if $z \leq y$ and $(y \ominus z)^\uparrow = x^\uparrow$.*

Proof. This is just Lemma 27 applied to the effect algebra $[0, x^\uparrow]_E$. $\square$

Corollary 28 shows that it is possible to express the mapping (M3) in terms of the triple. We now prove that we can construct an isomorphic copy of E from the triple. In our construction, we use only the mappings (M1)-(M3).

Theorem 29. *Let E be a complete lattice ordered effect algebra. Let $T(E)$ be a subset of $S(E) \times M(E)$ given by*

$$T(E) = \{\langle w_S, w_M\rangle : w_M \in h(w_S{}')\}.$$

Equip $T(E)$ with a relation $\leq$ given by $\langle u_S, u_M\rangle \leq \langle v_S, v_M\rangle$ if and only if $u_S \leq v_S$ and $u_M \ominus \pi_{h(v_S)}(u_M) \leq v_M$ and define a partial operation $\ominus$ with domain $\leq$ given by $\langle v_S, v_M\rangle \ominus \langle u_S, u_M\rangle = \langle w_S, w_M\rangle$, where

$$w_S = (v_S \ominus u_S) \ominus [\pi_{h(v_S)}(u_M)]^\uparrow$$

and

$$w_M = S(\pi_{h(v_S)}(u_M)) \oplus (v_M \ominus (u_M \ominus \pi_{h(v_S)}(u_M))).$$

Then $(T(E), \leq, \ominus)$ is a D-poset and the mapping $\phi : E \to T(E)$ given by $\phi(u) = \langle u^\downarrow, u \ominus u^\downarrow\rangle$ is an isomorphism of D-posets.

Proof. Obviously, ϕ is surjective. It remains to prove that $u \leq v$ if and only if $\phi(u) \leq \phi(v)$ and that $\phi(v \ominus u) = \phi(v) \ominus \phi(u)$.

Suppose that $u \leq v$. We write $\phi(u) = \langle u_S, u_M\rangle$ and $\phi(v) = \langle v_S, v_M\rangle$. Since $u \leq v$, there is a block $B \supseteq \{u, v\}$. By Corollary 14, $\{u^\downarrow, u^\uparrow, v^\downarrow, v^\uparrow\} \subseteq B$.

Obviously, $u_S \leq v_S$. We have

$$u = (u \ominus u^\downarrow) \oplus u^\downarrow = u_M \oplus u_S = [u_M \ominus (u_M \wedge v_S)] \oplus (u_M \wedge v_S) \oplus u_S$$

In particular,

$$u_M \ominus (u_M \wedge v_S) = u_M \ominus \pi_h(v_S)(u_M) \leq u \leq v = v_M \oplus v_S. \tag{4}$$

Since B is a sublattice and subeffect algebra of E, the values of all subexpressions occurring in (4) are elements of B and we may apply the Riesz decomposition property:

$$u_M \ominus (u_M \wedge v_S) = w_1 \oplus w_2,$$

where $w_1 \leq v_M$ and $w_2 \leq v_S$. Since B is an MV-effect algebra, $u_M \leftrightarrow v_S$, that means, $u_M \ominus (u_M \wedge v_S) \leq v_S{}'$. Since v_S is sharp, $w_2 = 0$. Therefore, $u_M \ominus (u_M \wedge v_S) = w_1 \leq v_M$.

Suppose that $u_S \leq v_S$ and that $u_M \ominus (u_M \wedge v_S) \leq v_M$. We need to prove that $u \leq v$. As

$$u = [u_M \ominus (u_M \wedge v_S)] \oplus (u_M \wedge v_S) \oplus u_S \tag{5}$$

and

$$v = v_S \oplus v_M = (v_S \ominus u_S) \oplus u_S \oplus v_M, \tag{6}$$

it suffices to prove that $u_M \wedge v_S \leq v_S \ominus u_S$.

We see that

$$\begin{aligned} u_M \wedge v_S = u_M \wedge [(v_S \ominus u_S) \oplus u_S] = u_M \wedge [(v_S \ominus u_S) \vee u_S] = \\ = [u_M \wedge (v_S \ominus u_S)] \vee (u_M \wedge u_S). \end{aligned}$$

However, $u_M \wedge u_S = 0$.

We have proved that $u \leq v$ if and only if $\phi(u) \leq \phi(v)$. This implies that ϕ is injective. Let us prove that ϕ preserver $\ominus$. Again, suppose that $u \leq v$. By (5) and (6),

$$v \ominus u = [(v_S \ominus u_S) \oplus v_M] \ominus ((u_M \wedge v_S) \oplus [u_M \ominus (u_M \wedge v_S)]). \tag{7}$$

Since $u \leq v$, $u_M \ominus (u_M \wedge v_S)]$. From (7),

$$u_M \wedge v_S \leq (v_S \ominus u_S) \oplus v_M.$$

By the Riesz decomposition property, there are $w_1 \leq v_S \ominus u_S$ and $w_2 \leq v_M$ such that $u_M \wedge v_S = w_1 \oplus w_2$. Since $v_S \wedge v_M = 0$, $w_2 = 0$. This implies that $u_M \wedge v_S \leq v_M$ and the equality (7) transforms to

$$v \ominus u = [(v_S \ominus u_S) \ominus (u_M \wedge v_S)] \oplus [v_M \ominus [u_M \ominus (u_M \wedge v_S)]]. \tag{8}$$

We note that the summands of the right-hand side of (8) are disjoint. As $(u_M \wedge v_S)^\uparrow \leq v_S \ominus u_S$, we can write

$$(v_S \ominus u_S) \ominus (u_M \wedge v_S) = [(v_S \ominus u_S) \ominus (u_M \wedge v_S)^\uparrow] \oplus [(u_M \wedge v_S)^\uparrow \ominus (u_M \wedge v_S)]. \tag{9}$$

Let us put

$$s = (v_S \ominus u_S) \ominus (u_M \wedge v_S)^\uparrow = (v_S \ominus u_S) \ominus [\pi_{h(v_S)}(u_M)]^\uparrow,$$

$$m_1 = (u_M \wedge v_S)^\uparrow \ominus (u_M \wedge v_S) = S(\pi_{h(v_S)}(u_M))$$

and

$$m_2 = v_M \ominus [u_M \ominus (u_M \wedge v_S)] = v_M \ominus (u_M \ominus \pi_{h(v^\uparrow)}(u_M)).$$

By (8) and (9), $v \ominus u = s \oplus m_1 \oplus m_2$. We see that $\phi(u) \ominus \phi(v) = \langle s, m_1 \oplus m_2 \rangle$. It remains to prove that $m_1 \oplus m_2$ is meager. Obviously, m_1 and m_2 are meager. Since $m_1 \leq v_S$ and $m_2 \leq v_M$, $m_1 \wedge m_2 = 0$. Therefore, $m_1 \oplus m_2 = m_1 \vee m_2$. By Lemma 23, $m_1 \vee m_2 \in M(E)$. □

6. Concluding remarks

It would be interesting to know in what classes of lattice ordered effect algebras Theorem 29 holds. The kernel of the present proof is (quite obviously) making use of the fact that in an orthocomplete homogeneous effect algebra we may reach the element $y \ominus y^\downarrow$ by summing up the elements which are below $y, y^\downarrow$. This technique does not work for all sharply dominating homogeneous effect algebras.

Example 30. Let E be the MV-effect algebra of all continuous functions $[0,1] \to [0,1]$, equipped with the usual addition of functions restricted to E. Note that $S(E) = \{0,1\}$. For the meager element f given by $f(x) = x$ we have $f^\downarrow = 0$, but $f \ominus f^\uparrow = f$ cannot be reached by summing up the functions from the set $\{g : g \leq f, f'\}$, because we have $g(1) = 0$ for every g.

Note that, in Example 30, we can overcome this difficulty by extending E to the (still incomplete) MV-effect algebra of all piecewise continuous functions $[0,1] \to [0,1]$. This shows that there is probably some class of triple-representable lattice ordered effect algebras strictly between the complete class and the sharply dominating class.

Another open question is whether it is possible to extend Theorem 29 to the class of orthocomplete homogeneous effect algebras. Here, the first principal difficulty we can see is Proposition 26, which is generally not true in an orthocomplete homogeneous effect algebra, since $h(a)$ needs not to be an ideal.

References

[1] M.K. Bennett and D.J. Foulis, 'Interval and scale effect algebras.' *Advances in Applied Mathematics* (1997) 200–215.

[2] G. Cattaneo, 'A unified framework for the algebra of unsharp quantum mechanics.' *International Journal of Theoretical Physics* 36 (1997) 3085–3117.

[3] C.C. Chang, 'Algebraic analysis of many-valued logics.' *Transactions of American Mathematical Society* 89 (1959) 74–80.

[4] C.C. Chen and G. Grätzer, 'Stone lattices I Construction theorems.' *Canad. J. Math.* 21 (1969) 884–894.

[5] C.C. Chen and G. Grätzer, 'Stone lattices II. Structure theorems.' *Canad. J. Math.* 21 (1969) 895–903.

[6] G. Chevalier and S.Pulmannová, 'Some ideal lattices in partial abelian monoids.' preprint, Mathematical Institute of Slovak Academy of Sciences, Bratislava, (1998).

[7] F. Chovanec and F. Kôpka, 'Boolean D-posets.' *Tatra Mountains Mathematical Publications* 10 (1997) 1–15.

[8] A. Dvurečenskij, 'Effect algebras which can be covered by MV-algebras.' *International Journal of Theoretical Physics* 41 (2002) 221–229.

[9] A. Dvurečenskij and M.G. Graziano, 'Remarks on representations of minimal clans.' *Tatra Mountains Mathematical Publications* 15 (1998) 31–53.

[10] A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures* (Kluwer, Dordrecht and Ister Science, Bratislava, 2000).

[11] D.J. Foulis and M.K. Bennett, 'Effect algebras and unsharp quantum logics.' *Foundations of Physics* 24 (1994) 1331–1352.

[12] D.J. Foulis, R. Greechie and G. Rütimann, 'Filters and supports in othoalgebras.' *International Journal of Theoretical Physics* 35 (1995) 789–802.

[13] R. Giuntini and H. Greuling, 'Toward a formal language for unsharp properties.' *Foundations of Physics* 19 (1989) 931–945.

[14] R. Greechie, D. Foulis and S. Pulmannová, 'The center of an effect algebra.' *Order* 12 (1995) 91–106.

[15] S. Gudder, 'Sharply dominating effect algebras.' *Tatra Mountains Mathematical Publications* 15 (1998) 23–30.

[16] G. Jenča, 'Subcentral ideals in generalized effect algebras.' *Intertnational Journal of Theoretical Physics* 39 (2000) 745–755.

[17] G. Jenča, 'Blocks of homogeneous effect algebras.' *Bulletin of Australian Mathematical Society* 64 (2001) 81–98.

[18] G. Jenča, 'Finite homogeneous and lattice ordered effect algebras.' *Discrete Mathematics* 272 (2003) 197–214.

[19] G. Jenča and S. Pulmannová, 'Quotients of partial abelian monoids and the Riesz decomposition property.' *Algebra universalis* 47 (2002) 443–477.

[20] G. Jenča and S. Pulmannová, 'Orthocomplete effect algebras.' *Proceedings of American Mathematical Society* 131 (2003) 2663–2671.

[21] G. Jenča and Z. Riečanová, 'On sharp elements in lattice ordered effect algebras.' *BUSEFAL* 80 (1999) 24–29.

[22] F. Kôpka and F. Chovanec, 'D-posets.' *Mathematica Slovaca* 44 (1994) 21–34.

[23] D. Mundici, 'Interpretation of AF C^*-algebras in Lukasziewicz sentential calculus.' *Journal of Functional Analysis* 65 (1986) 15–63.

[24] D. Mundici, 'MV-algebras are categorically equivalent to bounded commutative BCK-algebras.' *Mathematica Japonica* 6 (1986) 889–894.

[25] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics* (Kluwer, Dordrecht, 1991).

[26] Z. Riečanová, 'A generalization of blocks for lattice effect algebras.' *International Journal of Theoretical Physics* 39 (2000) 855–865.

[27] Z. Riečanová, 'Continuous lattice effect algebras admitting order continuous states.' *Fuzzy sets and systems* 136 (2003) 41–54.

[28] H. Yutani, 'The class of commutative BCK-algebras is equationally definable.' *Math. Seminar Notes* 5 (1977) 207–210.

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