The block structure of complete lattice ordered effect algebras

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Effect Algebras

Foulis and Bennett 1994; Kôpka and Chovanec 1994; Giuntini and Greuling 1989

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ satisfying the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.

(E4) If $a \oplus 1$ exists, then $a = 0$

Basic Relationships

Let E be an effect algebra.

- *Cancellativity:* $a \oplus b = a \oplus c \Rightarrow b = c$.
- *Partial difference:* If $a \oplus b = c$ then we write $a = c \ominus b$. $\ominus$ is well defined and $a' = 1 \ominus a$.
- *Poset:* Write $b \leq c$ iff $\exists a : a \oplus b = c$; $(E, \leq)$ is then a bounded poset.
- *Domain of $\oplus$:* $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$.

Subalgebras and morphisms

Definition

Let E be an effect algebra. A subset $F \subseteq E$ is *a subeffect algebra* of E iff

- $1 \in F$ and
- for all $a, b \in F$ such that $a \ominus b$ exists, $a \ominus b \in F$.

- If F is a subeffect algebra of E, then $0 \in F$ and F is closed with respect to $\oplus$ and the $'$ operations.

Definition

Let E, F be effect algebras, let $\phi : E \to F$. We say that ϕ is a *morphism of effect algebras* iff

- $\phi(1) = 1$ and
- for all $a, b \in E$ such that $a \oplus b$ exists in E, $\phi(a) \oplus \phi(b)$ exists in F and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$

Classes of Effect Algebras

- An effect algebra is an *orthomodular lattice* iff it is lattice ordered and, for all elements a, $a \wedge a' = 0$.
- A lattice-ordered effect algebra is an *MV-effect algebra* iff $a \wedge b = 0$ implies that $a \oplus b$ exists.

Sharp Elements

- An element a of an effect algebra is called *sharp* iff $a \wedge a' = 0$.
- We write $S(E)$ for the set of of all sharp elements of an effect algebra.
- (Jenča and Riečanová 1999) The set of all sharp elements of a lattice ordered effect algebra E forms an orthomodular lattice which is a subeffect algebra and a sublattice of E.

Blocks of Lattice Ordered Effect algebras

- (Riečanová 1999) Every lattice ordered effect algebra is a union of maximal sub-effect algebras which are MV-effect algebras.
- This result is a generalization of the well-known fact that every orthomodular lattice is a union of its blocks. Hence the following definition is natural.

Definition

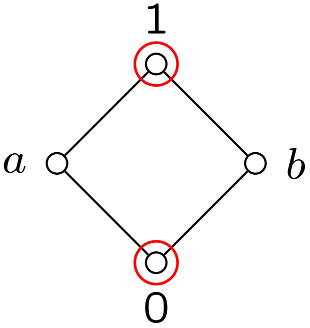
Let E be a lattice ordered effect algebra. A *block of* E is a maximal sub-effect algebra of E which is an MV-effect algebra.

- (Riečanová 1999) A subset M of a lattice ordered effect algebra is a block iff M is a maximal subset with respect to the compatibility condition

$$\forall a, b \in M : a \ominus (a \wedge b) = (a \vee b) \ominus b.$$

Example 1

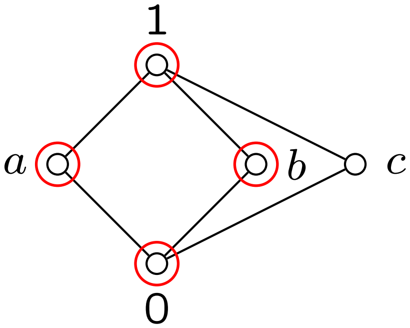
The diamond



$$a \oplus a = b \oplus b = 1$$

- $S(E)$ is a Boolean algebra, but E has two blocks.
- For any block B of E, $S(E) \cap B$ is a block of $S(E)$.

Example 2
Very simple



$$a \oplus b = c \oplus c = 1$$

- There are two blocks here, a Boolean algebra 2^2 and a 3-element chain C_3.
- We see that $C_3 \cap S(E) = \{0, 1\}$ is not a block of $S(E)$.

Blocks of E and blocks of $S(E)$

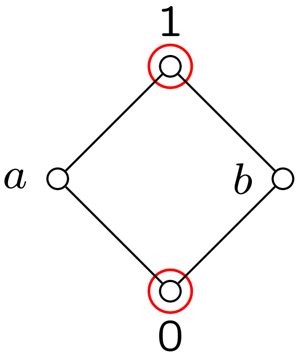
Every block of $S(E)$ is the center of some block of E

Theorem (Jenča and Riečanová 1999)

Let E be a lattice ordered effect algebra. B be a block of $S(E)$. Then there is a block M of E such that $M \cap S(E) = B$.

Connecting OMLs and finite Lattice Ordered EAs

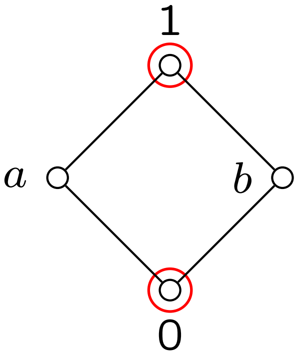
A simple example



$a \oplus a = b \oplus b = 1$

Connecting OMLs and finite Lattice Ordered EAs

A simple example



$a \oplus a = b \oplus b = 1$

$a_1 \oplus a_2 = b_1 \oplus b_2 = 1$

Connecting OMLs and finite Lattice Ordered EAs

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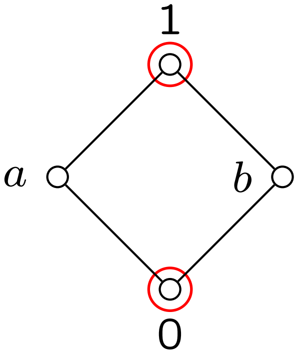
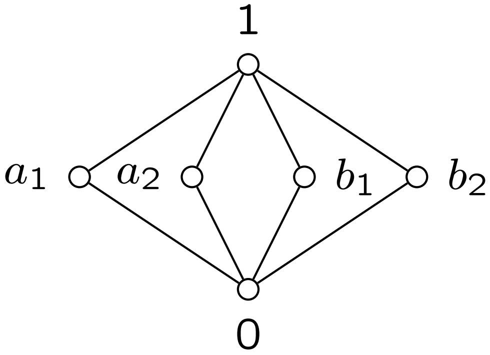


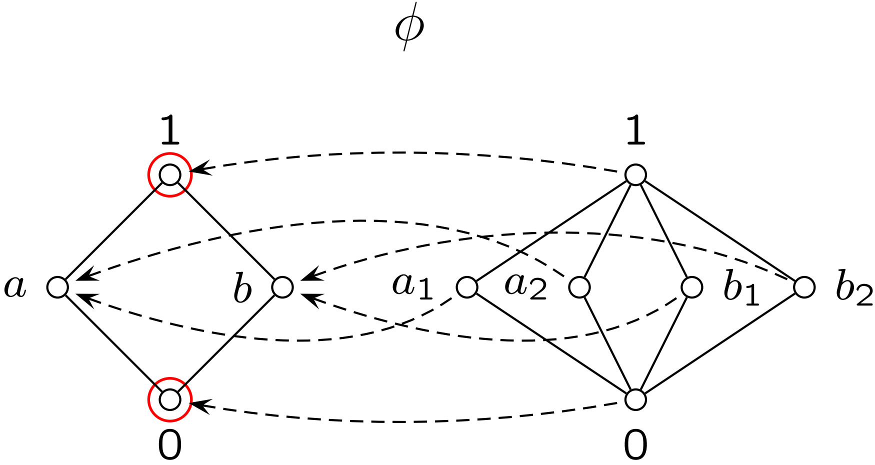


$a \oplus a = b \oplus b = 1$

$a_1 \oplus a_2 = b_1 \oplus b_2 = 1$

Connecting OMLs and finite Lattice Ordered EAs

A simple example



$a \oplus a = b \oplus b = 1$ $\qquad$ $a_1 \oplus a_2 = b_1 \oplus b_2 = 1$

Connecting OMLs and finite Lattice Ordered EAs

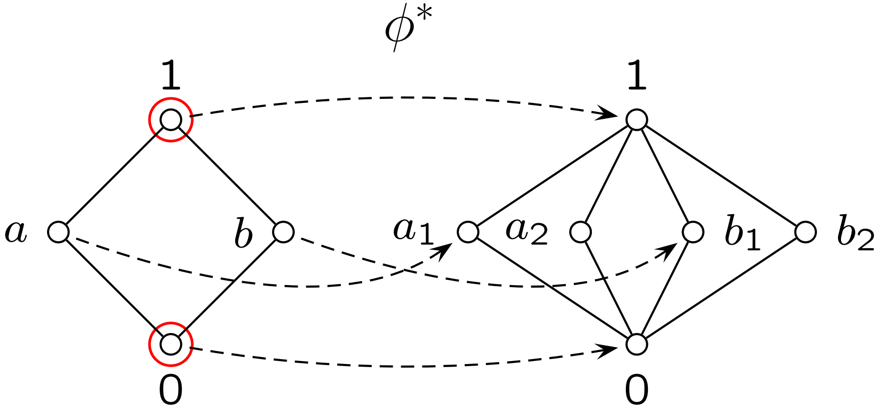
Theorem (Jenča 2003)

For every finite lattice ordered effect algebra E, there is a finite orthomodular lattice $O(E)$ and a surjective morphism of effect algebras $\phi : O(E) \to E$ such that

- *for every block B of $O(E)$, $\phi(B)$ is a block of E and*
- *for every block M of E, $\phi^{-1}(M)$ is a block of $O(E)$.*

Moreover (unpublished), there is a bounded injective lattice morphism $\phi^* : E \to O(E)$ such that, for all $x \in E$, $\phi(\phi^*(x)) = x$.

Connecting OMLs and finite Lattice Ordered EAs
The bounded lattice embedding



Connecting BAs and MV-effect algebras

R-generated Boolean algebras

Let L be a bounded distributive lattice. Recall, that a *Boolean algebra R-generated by* L is a Boolean algebra $B(L)$ such that

- L is a $0, 1$-sublattice of $B(L)$ and
- L generates $B(L)$, as a Boolean algebra.

These properties determine $B(L)$, up to isomorphism.

Theorem

For every MV-effect algebra M there is a surjective morphism of effect algebras $\phi : B(M) \to M$ *and a bounded lattice embedding* $\phi^* : M \to B(M)$ *such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\phi^*} & B(M) \\ & {}_{\mathrm{id}_M}\searrow & \downarrow \phi \\ & & M \end{array}$$

commutes.

Example - the Real Unit Interval

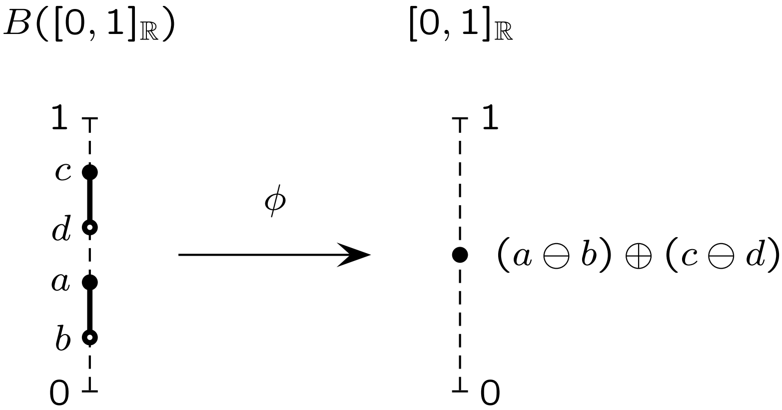
The Boolean algebra R-generated by $[0,1]_{\mathbb{R}}$

For the real unit interval $[0,1]_{\mathbb{R}}$, the Boolean algebra $B([0,1]_{\mathbb{R}})$ R-generated by $[0,1]_{\mathbb{R}}$ is the Boolean algebra of subsets of $[0,1]_{\mathbb{R}}$ of the form

$$(b_1, a_1] \dot{\cup} (b_2, a_2] \dot{\cup} \ldots \dot{\cup} (b_n, a_n].$$

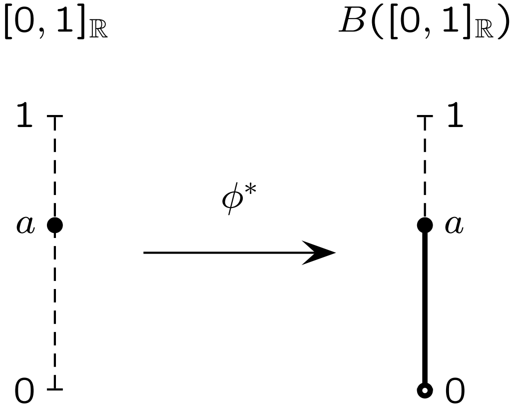
Example - the Real Unit Interval

The ϕ map



Example - the Real Unit Interval

The ϕ^* map



Sharp covers and sharp kernels
The definitions

Let E be a complete lattice ordered effect algebra, let $x \in E$.
We denote

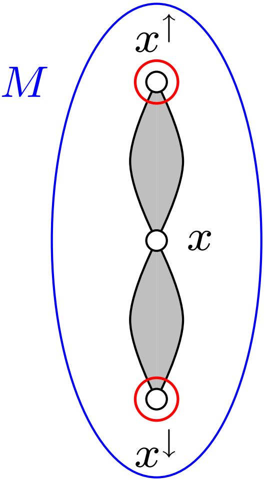
- $x^{\uparrow}$ for the smallest sharp element above x and
- $x^{\downarrow}$ for the greatest sharp element below x, that means,

$$x^{\uparrow} = \bigwedge\{y \in S(E) : y \geq x\}$$
$$x^{\downarrow} = \bigvee\{y \in S(E) : y \leq x\}.$$

- $x^{\uparrow}$ is called the *sharp cover of* x.
- $x^{\downarrow}$ is called the *sharp kernel of* x.

Sharp covers and sharp kernels

The properties



Theorem (Jenča 2004, submitted to AU)

Let E be a complete lattice ordered effect algebra, let $x \in E$. Pick a block M of E with $x \in M$. Then $[x^{\downarrow}, x] \cup [x, x^{\uparrow}] \subseteq M$.

Corollary

Let E be a complete lattice ordered effect algebra, let $x \in E$ be such that $x^{\downarrow} = 0$. Then $[0, x]$ is an MV-effect algebra.

Theorem (Pulmannová 2005)

Let E be a σ-complete effect algebra, let $x \in E$. Pick a block M of E with $x \in M$. Then $x^{\uparrow}, x^{\downarrow}$ exist and belong to M.

Problem

Let E be a σ-complete lattice ordered effect algebra, let $x \in E$. Pick a block M of E with $x \in M$. Is it true that $[x^{\downarrow}, x] \cup [x, x^{\uparrow}] \subseteq M$?

Theorem (Jenča 2005, to appear in JAustMS)

For every complete lattice ordered effect algebra E, there is a orthomodular lattice $O(E)$ and a surjective morphism of effect algebras $\phi : O(E) \to E$ such that

- *for every block B of $O(E)$, $\phi(B)$ is a block of E and*
- *for every block M of E, $\phi^{-1}(M)$ is a block of $O(E)$.*

Moreover, there is a bounded injective lattice morphism $\phi^* : E \to O(E)$ such that, for all $x \in E$, $\phi(\phi^*(x)) = x$.

From now on, E is a complete lattice ordered effect algebra.

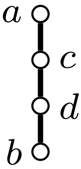
Definition

Let a/b denote an ordered pair of elements satisfying $a \geq b$. We say that a/b is *a quotient of* E.

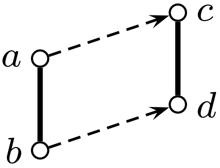
- The set of all quotients of E is denoted by $Q(E)$.
- We denote $|a/b| = a \ominus b$ (*the size of* a/b).

Connecting OMLs and Complete Lattice Ordered EAs

The relations $\nearrow$, $\searrow$, and $\sqsubseteq$



- Let us write $c/d \sqsubseteq a/b$ (or $a/b \sqsupseteq c/d$) if and only if $b \leq d \leq c \leq a$.

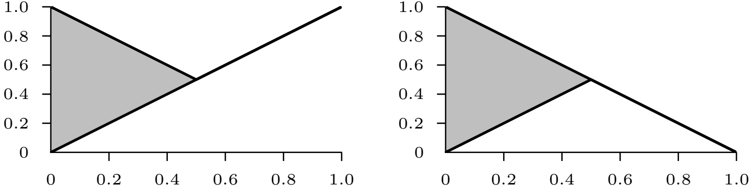


- Let us write $a/b \nearrow c/d$ if and only if
  - $a \leq c$, and
  - $b \leq d$, and
  - $c \ominus a = d \ominus b$, and
  - $(c \ominus a) \wedge (a \ominus b) = 0$.

- Note that $a/b \nearrow c/d$ implies that $a \ominus b = c \ominus d$.
- $\nearrow$ is transitive.This is not true in a general effect algebra.
- Let us write $\searrow$ for the inverse relation of $\nearrow$.

Connecting OMLs and Complete Lattice Ordered EAs

An example of $\nearrow$ and $\searrow$ in $[0, 1]^{[0,1]}$



$a/b \searrow c/d$ in $[0, 1]^{[0,1]}$

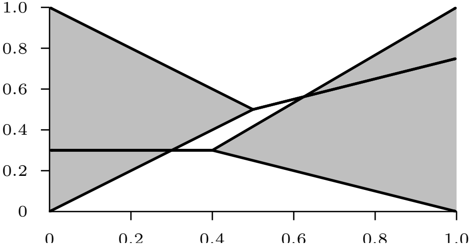
Connecting OMLs and Complete Lattice Ordered EAs
Disjoint quotients

- Let us write $\equiv$ for the transitive closure of $\nearrow \cup \searrow$.
- We say that quotients a/b and c/d are *disjoint* if and only if for all x/y, z/w

$$a/b \sqsupseteq x/y \equiv z/w \sqsubseteq c/d \implies x = y.$$

Connecting OMLs and Complete Lattice Ordered EAs

An example of disjoint quotients in $[0,1]^{[0,1]}$



Disjoint quotients in $[0,1]^{[0,1]}$.

Connecting OMLs and Complete Lattice Ordered EAs

Orthogonal sets of quotients

- We say that a finite set of quotients

$$\mathbf{f} = \{a_1/b_1, \ldots, a_n/b_n\}$$

  is *orthogonal* if and only if
  - $\mathbf{f}$ is pairwise disjoint and
  - the sum
  $$|\mathbf{f}| := |a_1/b_1| \oplus \cdots \oplus |a_n/b_n|$$
    exists in E.
- A finite set of quotients $\mathbf{t}$ is *a test* if and only if
  - $\mathbf{t}$ is orthogonal and
  - $|\mathbf{t}| = 1$.

Let X be a nonempty set, let $\mathcal{N}, \mathcal{T} \subseteq 2^X$. We say that a triple $(X, \mathcal{T}, \mathcal{N})$ is a *generalized test space* if and only if the following conditions are satisfied.

(GTS1) $X = \bigcup_{\mathbf{t} \in \mathcal{T}} \mathbf{t}$.

(GTS2) $\mathcal{N}$ is an ideal of 2^X, that is, $\mathcal{N}$ is nonempty and for all $\mathbf{o}_1, \mathbf{o}_2 \subseteq X$ we have $\mathbf{o}_1 \cup \mathbf{o}_2 \in \mathcal{N}$ if and only if $\mathbf{o}_1, \mathbf{o}_2 \in \mathcal{N}$.

(GTS3) For all $\mathbf{t}_1 \subseteq \mathbf{t}_2 \subseteq X$ such that $\mathbf{t}_1 \in \mathcal{T}$, we have $\mathbf{t}_2 \in \mathcal{T}$ if and only if $\mathbf{t}_2 \setminus \mathbf{t}_1 \in \mathcal{N}$.

(GTS4) For all $\mathbf{t}_1 \subseteq \mathbf{t}_2 \subseteq X$ such that $\mathbf{t}_2 \setminus \mathbf{t}_1 \in \mathcal{N}$, we have $\mathbf{t}_1 \in \mathcal{T}$ if and only if $\mathbf{t}_2 \in \mathcal{T}$.

- $\mathcal{T}_E$ is the set of all tests.
- A finite set of quotients **f** is *an event* if and only if $\mathbf{f} \subseteq \mathbf{t}$ for some test **t**.
- A finite set of quotients **o** is *a null event* if and only if **o** contains only quotients of the type x/x.
- $\mathcal{N}_E$ is the set of all null events.
- $\Omega(E) := (Q(E), \mathcal{T}_E, \mathcal{N}_E)$ is then a generalized test space.

Two events $\mathbf{f}, \mathbf{g}$ are

- *Orthogonal* (in symbols $\mathbf{f} \perp \mathbf{g}$) iff
  - $\mathbf{f} \cup \mathbf{g}$ is an event, and
  - $\mathbf{f} \cap \mathbf{g}$ is a null event.
- *Local complements* (in symbols $\mathbf{f} \operatorname{loc} \mathbf{g}$) iff
  - $\mathbf{f}$ and $\mathbf{g}$ are orthogonal, and
  - $\mathbf{f} \cup \mathbf{g}$ is a test.
- *Perspective* (in symbols $\mathbf{f} \sim \mathbf{g}$) iff they share a local complement.

The generalized test space $\Omega(E)$ is algebraic, that means:

- for all events $\mathbf{f}, \mathbf{g}, \mathbf{h}$

$$(\mathbf{f} \sim \mathbf{g}) \text{ and } (\mathbf{g} \operatorname{loc} \mathbf{h}) \Longrightarrow \mathbf{f} \operatorname{loc} \mathbf{h}.$$

Consequences:

- $\sim$ is an equivalence relation,
- $\sim$ preserves the union of orthogonal events.

Connecting OMLs and Complete Lattice Ordered EAs

The construction of $O(E)$

$O(E)$ is constructed as follows.

- $O(E)$ is the set of all equivalence classes of events with respect to the $\sim$ relation.
- The unit element of $O(E)$ is the set of all tests.
- The zero element of $O(E)$ is the set of all events that contain only the elements of the type x/x (the *null events*).
- The partial $\oplus$ operation on $O(E)$ is given by the rule

$$[\mathbf{f}]_\sim \oplus [\mathbf{g}] = [\mathbf{f} \cup \mathbf{g}]_\sim$$

  whenever **f** and **g** are orthogonal events.

$O(E)$ is then a lattice ordered orthoalgebra, that is, an orthomodular lattice.

Connecting OMLs and Complete Lattice Ordered EAs

The construction of ϕ and ϕ^*

- $\phi : O(E) \to E$ is given by the rule

$$\phi([\mathbf{f}]_{\sim}) = |\mathbf{f}|.$$

- $\phi^* : E \to O(E)$ is given by the rule

$$\phi^*(a) = \{a/0\}.$$

Some other facts

- If E is an MV-effect algebra, then $O(E)$ is isomorphic to the Boolean algebra R-generated by E.
- E is an OML if and only if $E \simeq O(E)$.
- An element a of E is sharp iff $\phi^{-1}(a)$ is a singleton.
- $\phi^{-1}(S(E))$ is a sub-orthomodular lattice of $O(E)$.

Theorem (Jenča 2005, to appear in JAustMS)

For every complete lattice ordered effect algebra E, there is a orthomodular lattice $O(E)$ and a surjective morphism of effect algebras $\phi : O(E) \to E$ such that

- *for every block B of $O(E)$, $\phi(B)$ is a block of E and*
- *for every block M of E, $\phi^{-1}(M)$ is a block of $O(E)$.*

Moreover, there is a bounded injective lattice morphism $\phi^* : E \to O(E)$ such that, for all $x \in E$, $\phi(\phi^*(x)) = x$.

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http://matika.elf.stuba.sk/KMAT/GejzaJenca

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