COMPATIBILITY MAPPINGS IN INTERVAL EFFECT ALGEBRAS

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1. Setup

1.1. **Partially ordered abelian groups.** Let G be an (additive) abelian group. We say that G is a *partially ordered abelian group* iff G is equipped with a partial order that is compatible with addition, that means, for all $a, b, t \in G$,

$$a \geq b \Longrightarrow a + t \geq b + t.$$

[1]

For a partially ordered abelian group G, we write

$$G^+ = \{a \in G : a \geq 0\}.$$

The elements of G^+ are called *positive*. Obviuosly, G^+ is a submonoid of G. Moreover, G^+ is *conical*, that means, if $a, b \in G^+$ and $a + b = 0$, then $a = b = 0$.

Given a group G, it is easy to see that there is a one-to one correspondence between partial orders on G and conical submonoids of G.

1.2. **Order units.** Let G be a partially ordered abelian group. We say that $u \in G^+$ is an order unit iff for every $a \in G$ there is $n \in \mathbb{N}$ such that $n.u \geq a$.

A pair (G, u), where G is a partially ordered abelian group and u is an order unit of G is called *a unital group*.

Let (G_1, u_1), (G_2, u_2) be unital groups. A mapping $\phi : G_1 \to G_2$ is a *morphism of unital groups* iff ϕ is a group homomorphism, $x \geq y$ implies $\phi(x) \geq \phi(y)$ and $\phi(u_1) = u_2$.

For a morphism of unital groups, we write $\phi : (G_1, u_1) \to (G_2, u_2)$.

1.3. **Interval effect algebras.** One can construct examples of effect algebras from an arbitrary partially ordered abelian group $(G, \leq)$ in the following way: Choose any positive $u \in G$; then, for $0 \leq a, b \leq u$, define $a \oplus b$ if and only if $a + b \leq u$ and put $a \oplus b = a + b$. With such partial operation $\oplus$, the interval $[0, u]$ becomes an effect algebra $([0, u], \oplus, 0, u)$. Effect algebras which arise from partially ordered abelian groups in this way are called *interval effect algebras*, see [**?**].

1.4. **Group valued measures and ambient groups.** Let E be an effect algebra and let (G_2, u_2) be a unital group. A morphism of effect algebras from E to the interval effect algebra $[0, u_2]_{G_2}$ is called a *group-valued measure*.

⟨prop:ambient⟩ **Proposition 1.** [2] [**?**] *Let E be an interval effect algebra. There exists a unital group (G_1, u_1) such that $E = [0, u_1]_{G_1}$, E generated G_1 and for every unital group (G_2, u_2) and every group valued measure $\phi : E \to [0, u_2]_{G_2}$, phi extends to a unique*

[1] Give some easy examples, and $\mathcal{S}(\mathcal{H})$

[2] Look it up in the original paper.

morphism of unital groups $\widehat{\phi} : (G_1, u_1) \to (G_2, u_2)$*. The unital group* (G_1, u_1) *is unique, up to isomorphism.*

The unital group (G_1, u_1) from Proposition 1 is called *the ambient group* of E, denoted by $G(E)$.

1.5. **Möbius inversion theorem.** We say that a partially ordered set $(P, \leq)$ is *locally finite* if and only if every closed interval

$$[x, y]_P := \{z \in P : x \leq z \leq y\}$$

is a finite set.

Let G be an abelian group and let $(P, \leq)$ be a locally finite partially ordered set. Define $I(P)$ to be the set of all pairs $(x, y) \in P \times P$ such that $(x \leq y)$.

There exists a unique function $\mu : I(P) \to G$ such that, for all $(x, y) \in I(P)$,

(1) eq:mobius
$$\sum_{x \leq z \leq y} \mu(x, z) = \delta_{x,y},$$

where $\delta_{x,y}$ is the Kronecker delta. We say the μ is the *Möbius map of the poset* P.

To see that the Möbius map exists and is unique, observe that the equation 1 allows for an inductive definition of μ.

Indeed, let P be a locally finite poset. For $(x, y) \in I(P)$, let us write $h(x, y)$ for the height of the interval $[x, y]_P$. If $h(x, y) = 0$, then $x = y$ and $\mu(x, y) = \delta_{x,y} = 1$. Let $(x, y) \in I(P)$, $h(x, y) = n > 0$ and suppose that we already know the values $\mu(x, z)$ for all $(x, z) \in I(P)$ with $h(x, z) < n$.

Since $h(x, y) > 0$, $x \neq y$. Therefore, by equation 1,

$$\sum_{x \leq z \leq y} \mu(x, z) = \delta_{x,y} = 0.$$

This implies that

$$\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z),$$

and the values of $\mu(x, z)$ are already known.

Example 1. Let S be a set, write $Fin(S)$ for the set of all finite subsets of S. For the poset $(Fin(S), \subseteq)$, we have $\mu(X, Z) = (-1)^{|X|+|Z|}$.

Theorem 1 (Möbius inversion formula)**.** *Let* $f : I(P) \to A$*, define* $f^{\leq}(x, y) := \sum_{x \leq z \leq y} f(x, z)$*. Then*

$$f(x, y) = \sum_{x \leq z \leq y} \mu(x, z) f^{\leq}(z, y).$$

We say that $f(x, y)$ is the Möbius inversion of $f^{\leq}$. [3]

[3] Look it up. What is the MI of what?

1.6. **Compatibility maps.** Let E be an interval effect algebra. Let $S \subseteq E$. Let us write $Fin(S)$ for the set of all finite subsets of S. Obviously, $(Fin(S), \subseteq)$ is a locally finite poset.

For every mapping $\alpha : Fin(S) \to G(E)$, we define a mapping $D_\alpha : I(Fin(S)) \to G$. For $(X, A) \in I(Fin(S))$, the value $D_\alpha(X, A) \in G$ is given by the rule

$$D_\alpha(X, A) := \sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|} \alpha(Z).$$

Note that there is an obvious connection to Möbius inversions: define $\hat{\alpha} : I(Fin(S)) \to G$ by

$$\hat{\alpha}(X, A) = \alpha(X).$$

Then D_α is the Möbius inversion of $\hat{\alpha}$. By the Möbius inversion formula we see that

$$\alpha(X) = \hat{\alpha}(X, A) = \sum_{X \leq Z \leq A} D_\alpha(X, Z),$$

for any $A \supseteq X$. In particular, $A := X$ yields $\alpha(X) = D_\alpha(X, X)$.

⟨lemma:formal⟩ **Lemma 1.** *Let E be an interval effect algebra. Let S be a subset of E, let $\alpha : Fin(S) \to G(E)$. For all $c \in S \setminus A$,*

$$D_\alpha(X, A) = D_\alpha(X, A \cup \{c\}) + D_\alpha(X \cup \{c\}, A \cup \{c\}).$$

Proof. [4] Let us rewrite

$$D_\alpha(X, A \cup \{c\}) = \sum_{X \subseteq Z \subseteq A \cup \{c\}} (-1)^{|X|+|Z|} \alpha(Z).$$

For any Z in the above sum, either $c \in Z$ or $c \notin Z$. If $c \in Z$, then $X \cup \{c\} \subseteq Z \subseteq A \cup \{c\}$. If $c \notin Z$, then $X \subseteq Z \subseteq A$. Consequently,

$$\begin{aligned} D_\alpha(X, A \cup \{c\}) = &\sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|} \alpha(Z) + \sum_{X \cup \{c\} \subseteq Z \subseteq A \cup \{c\}} (-1)^{|X|+|Z|} \alpha(Z) = \\ = &D_\alpha(X, A) + \sum_{X \cup \{c\} \subseteq Z \subseteq A \cup \{c\}} (-1)^{|X|+|Z|} \alpha(Z) \end{aligned}$$

It remains to observe that

$$\begin{aligned} &\sum_{X \cup \{c\} \subseteq Z \subseteq A \cup \{c\}} (-1)^{|X|+|Z|} \alpha(Z) = \\ = -&\sum_{X \cup \{c\} \subseteq Z \subseteq A \cup \{c\}} (-1)^{|X \cup \{c\}|+|Z|} \alpha(Z) = D_\alpha(X \cup \{c\}, A \cup \{c\}). \end{aligned}$$

□

?⟨def:cm⟩? **Definition 1.** Let E be an interval effect algebra, let $S \subseteq E$.

We say that a mapping $\alpha : Fin(S) \to E$ is a *compatibility mapping for S* if and only if the following conditions are satisfied.

(A1) $\alpha(\emptyset) = 1$,
(A2) for all $c \in S$, $\alpha(\{c\}) = c$,
(A3) for all (X, A) in $I(Fin(S))$, $0 \leq D_\alpha(X, A) \leq u$.

[4] The sums look awkward.

1.7. **Properties of compatibility maps.** To shorten our formulations, let us introduce some running notation:

- E is an interval effect algebra,
- S is a subset of E,
- $\alpha : Fin(S) \to E$ is a compatibility map.

⟨prop:base⟩ **Proposition 2.** *For all $A \in Fin(S)$ and $c \in S \setminus A$, $D_\alpha(X, A \cup \{c\}) \perp D_\alpha(X \cup \{c\}, A \cup \{c\})$ and*

$$D_\alpha(X, A) = D_\alpha(X, A \cup \{c\}) \oplus D_\alpha(X \cup \{c\}, A \cup \{c\}).$$

Proof. By Lemma 1. □

⟨coro:antitone⟩ **Corollary 1.** *α is an antitone map from $(Fin(S), \subseteq)$ to $(E, \leq)$.*

Proof. Let us prove that for any $c \in S \setminus X$, $\alpha(X \cup \{c\}) \leq \alpha(X)$. Put $X = A$ in Proposition 2 to obtain

$$\alpha(X) = D_\alpha(X, X) = D_\alpha(X, X \cup \{c\}) \oplus D_\alpha(X \cup \{c\}, X \cup \{c\}) \geq D_\alpha(X \cup \{c\}, X \cup \{c\}) = \alpha(X \cup \{c\}).$$

The rest of the proof is a trivial induction. □

⟨coro:lbound⟩ **Corollary 2.** *$\alpha(X)$ is a lower bound of X.*

Proof. Let $c \in X$. By Corollary 1, $\{c\} \subseteq X$ implies that

$$c = \alpha(\{c\}) \geq \alpha(X).$$

□

Corollary 3. *If $0 \in X$, then $\alpha(X) = 0$.*

Proof. Trivial, by Corollary 2. □

⟨coro:zero⟩ **Corollary 4.** *If $1 \notin X$, then $D_\alpha(X, X \cup \{1\}) = 0$.*

Proof. (By induction with respect to $|X|$.) If $X = \emptyset$, then

$$D_\alpha(X, X \cup \{1\}) = D_\alpha(\emptyset, \{1\}) = \alpha(\emptyset) - \alpha(\{1\}) = 1 - 1 = 0.$$

Suppose that the Corollary is true for some X and let $c \notin X$, $c \neq 1$. We want to prove that $D_\alpha(X \cup \{c\}, X \cup \{c\} \cup \{1\}) = 0$. Putting $A = X \cup \{1\}$ in Proposition 2 yields

$$D_\alpha(X, X \cup \{1\}) = D_\alpha(X, X \cup \{c\} \cup \{1\}) \oplus D_\alpha(X \cup \{c\}, X \cup \{c\} \cup \{1\}).$$

By the induction hypothesis, $D_\alpha(X, X \cup \{1\}) = 0$, and since

$$D_\alpha(X, X \cup \{1\}) \geq D_\alpha(X \cup \{c\}, X \cup \{c\} \cup \{1\}),$$

we may conclude that $D_\alpha(X \cup \{c\}, X \cup \{c\} \cup \{1\}) = 0$. □

Corollary 5. $\alpha(X) = \alpha(X \cup \{1\})$

Proof. If $1 \in X$, there is nothing to prove.

Suppose that $1 \notin X$. Putting $A = X$ and $c = 1$ in Proposition 2 yields

$$D_\alpha(X, X) = D_\alpha(X, X \cup \{1\}) \oplus D_\alpha(X \cup \{1\}, X \cup \{1\}).$$

By Corollary 4, $D_\alpha(X, X \cup \{1\}) = 0$, hence

$$\alpha(X) = D_\alpha(X, X) = D_\alpha(X \cup \{1\}, X \cup \{1\}) = \alpha(X \cup \{1\}).$$

□

1.8. **Examples of compatibility maps.**

⟨prop:Dwedge⟩ **Proposition 3.** *Let M be an MV-effect algebra. For the mapping $\bigwedge : Fin(M) \to M$,*

$$D_{\wedge}(X,A) = \bigwedge X \ominus ((\bigwedge X) \wedge (\bigvee A/X)).$$

Proof. The proof goes by induction with respect to $|A \setminus X|$.

If $|A \setminus X| = 0$, then $A = X$ and

$$D_{\wedge}(X,A) = D(X,X) = \bigwedge X.$$

For the right-hand side,

$$\bigwedge X \ominus ((\bigwedge X) \wedge (\bigvee A/X)) = \bigwedge X \ominus ((\bigwedge X) \wedge 0) = \bigwedge X.$$

Let $n \in \mathbb{N}$, suppose that the Proposition is true for all pairs (X,A) with $|A \setminus X| \leq n$. Let $X, A_1 \in Fin(M)$ be such that $X \subseteq A_1$ and $|A_1 \setminus X| = n+1$. Pick $c \in A_1 \setminus X$ and put $A := A_1 \setminus \{c\}$. Then $c \notin A$ and $A_1 = A \cup \{c\}$.

By Lemma 1,

$$D_{\wedge}(X,A) = D_{\wedge}(X, A \cup \{c\}) + D_{\wedge}(X \cup \{c\}, A \cup \{c\}),$$

hence

$$D_{\wedge}(X, A \cup \{c\}) = D_{\wedge}(X,A) - D_{\wedge}(X \cup \{c\}, A \cup \{c\}).$$

To abbreviate, let us write $x := \bigwedge X$, $a := \bigvee X \setminus A$. Note that $\bigvee(A \cup \{c\}) \setminus (X \cup \{c\}) = a$ and that $\bigvee(A \cup \{c\}) \setminus X = a \vee c$. We need to prove that

$$D_{\wedge}(X, A \cup \{c\}) = x \ominus x \wedge (a \vee c).$$

By the induction hypothesis, we may write

$$\begin{aligned} D(X,A) =& x \ominus x \wedge a \\ D_{\wedge}(X \cup \{c\}, A \cup \{c\}) =& (x \wedge c) \ominus (x \wedge c \wedge a), \end{aligned}$$

therefore

$$D_{\wedge}(X, A \cup \{c\}) = (x \ominus x \wedge a) - ((x \wedge c) \ominus (x \wedge c \wedge a)).$$

Thus, it remains to prove that

$$x \ominus x \wedge (a \vee c) = (x \ominus x \wedge a) - ((x \wedge c) \ominus (x \wedge c \wedge a)),$$

that means,

$$((x \wedge c) \ominus (x \wedge c \wedge a)) + x \ominus x \wedge (a \vee c) = x \ominus x \wedge a.$$

Since M is an MV-algebra, we may compute

$$\begin{aligned} ((x \wedge c) \ominus (x \wedge c \wedge a)) = ((x \wedge c) \ominus ((x \wedge c) \wedge (x \wedge a))) = ((x \wedge c) \vee (x \wedge a)) \ominus (x \wedge a) = \\ = x \wedge (c \vee a) \ominus (x \wedge a) = x \wedge (a \vee c) \ominus (x \wedge a), \end{aligned}$$

hence

$$\begin{gathered} ((x \wedge c) \ominus (x \wedge c \wedge a)) + x \ominus x \wedge (a \vee c) = \\ (x \wedge (a \vee c) \ominus (x \wedge a)) + (x \ominus x \wedge (a \vee c)) = x \ominus x \wedge a. \end{gathered}$$

□

Corollary 6. *Let G be an abelian l-group. The mapping $\bigwedge : Fin(M) \to M$ is a compatibility map.*

Proof. Clearly, the conditions (A1) and (A2) are satisfied.

Moreover, for any $(X, A) \in Fin(M)$, $u \geq \bigwedge X \geq ((\bigwedge X) \wedge (\bigvee A/X))$. Therefore, by Proposition 3, $u \geq D(X, A) \geq 0$ and we see that (A3) is satisfied. □

⟨prop:Dprod⟩ **Proposition 4.** *Let E be an interval effect algebra. Assume that $G(E)$ can be equipped with a product so that $(G(E), +, ., 0, 1, \leq)$ is a partially ordered commutative ring.*

Let $\Pi : Fin(E) \to E$ be given by

$$\Pi(\{x_1, \dots, x_n\}) = x_1 . \dots . x_n.$$

For every $(X, A) \in I(Fin(E))$ and $c \in E \setminus A$,

$$D_\Pi(X \cup \{c\}, A \cup \{c\}) = c.D_\Pi(X, A).$$

Proof. Let us compute

$$D_\Pi(X \cup \{c\}, A \cup \{c\}) = \sum_{X\cup\{c\}\subseteq Z\subseteq A\cup\{c\}} (-1)^{|X\cup\{c\}|+|Z|}\Pi(Z) =$$

$$\sum_{X\subseteq Y\subseteq A} (-1)^{|X\cup\{c\}|+|Y\cup\{c\}|}\Pi(Y \cup \{c\}) = \sum_{X\subseteq Y\subseteq A} (-1)^{|X|+|Y|}\Pi(Y \cup \{c\}) =$$

$$c. \sum_{X\subseteq Y\subseteq A} (-1)^{|X|+|Y|}\Pi(Y) = c.D(X, A).$$

□

Corollary 7. Π *is a compatibility map.*

Proof. The proof goes by induction with respect to $|A \setminus X|$.

For $A = X$, $D_\Pi(X, A) = D_\Pi(X, X) = \Pi(X)$ and $0 \leq \Pi(X) \leq 1$.

Let $n \in \mathbb{N}$. Suppose that, for all $A, X \in Fin(E)$ such that $|A \setminus X| = n$, $0 \leq D_\Pi(X, A) \leq 1$. Let $A_1, X \in Fin(E)$ be such that $|A_1 \setminus X| = n + 1$. Pick $c \in A_1 \setminus X$ and write $A = A_1 \setminus \{c\}$. We see that $c \notin A$ and that $A_1 = A \cup \{c\}$. We shall prove that $0 \leq D_\Pi(X, A \cup \{c\}) \leq 1$.

By Lemma 1 and Proposition 4,

$$D_\Pi(X, A \cup \{c\}) = D_\Pi(X, A) - D_\Pi(X \cup \{c\}, A \cup \{c\}) =$$
$$D_\Pi(X, A) - c.D_\Pi(X, A) = (1 - c).D_\Pi(X, A).$$

By the induction hypothesis, $0 \leq D_\Pi(X, A) \leq 1$, hence

$$0 \leq (1 - c).D_\Pi(X, A) \leq 1.$$

[5] □

Proposition 5. *Let E_1, E_2 be interval effect algebras. Let $\phi : E_1 \to E_2$ be a momorphism of effect algebras. If $S_1 \subseteq E_1$ is such that there is a compatibility mapping α_1 of S_1, then $\phi(S_1)$ admits a compatibility mapping.*

Proof. The mapping ϕ is a $G(E_2)$ valued measure on E_1. Therefore, there is a morphism of unigroups $\widehat{\phi} : (G(E_1), 1) \to (G(E_2), 1))$ extending ϕ.

For every $a \in \widehat{\phi}(S_1)$, fix $p(a) \in S_1$ such that $\widehat{\phi}(p(a)) = a$. Define $\alpha_2 : Fin(\widehat{\phi}(S_1)) \to E_2$ as follows:

$$\alpha_2(\{x_1, \dots, x_n\}) = \widehat{\phi}(\alpha_1(\{p(x_1), \dots, p(x_n)\})),$$

[5]Do we need 1 to be a ring unit? Maybe a general order unit u would do.

or, in other words, for $X \in Fin(S_2)$, $\alpha_2(X) = \widehat{\phi}(\alpha_1(p(X)))$. Then α_2 is a compatibility map for $\widehat{\phi}(S_1)$.

Indeed, condition (A1) and (A2) are trivially satisfied. For the proof of (A3) we may compute

$$D_{\alpha_2}(X, A) = \sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|}\alpha_2(Z) = \sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|}\widehat{\phi}(\alpha_1(p(Z))) =$$
$$\widehat{\phi}(\sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|}\alpha_1(p(Z))) = \widehat{\phi}(\sum_{p(X) \subseteq Y \subseteq p(A)} (-1)^{|p(X)|+|Y|}\alpha_1(Y)).$$

Since α_1 is a compatibility map, $(-1)^{|p(X)|+|Y|}\alpha_1(Y) \in E_1$. Therefore,

$$\widehat{\phi}(\sum_{p(X) \subseteq Y \subseteq p(A)} (-1)^{|p(X)|+|Y|}\alpha_1(Y)) \in E_2.$$

□

Corollary 8. *Let $(G, \leq)$ be a partially ordered abelian group with an order unit u, let M be an MV-algebra. Let $\phi : M \to [0, u]_G$ be a morphism of effect algebras. Then $\phi(M)$ admits a compatibility mapping.*

Proof. According to Mundici's theorem, the mapping □

⟨lemma:second⟩ **Lemma 2.** *Let $C, A, X \in Fin(S)$ be such that $X \subseteq A$ and $C \cap A = \emptyset$. Then $(D_\alpha(X \cup Y, A \cup C))_{Y \subseteq C}$ is an orthogonal family and*

$$\bigoplus_{Y \subseteq C} D_\alpha(X \cup Y, A \cup C) = D_\alpha(X, A).$$

Proof. The proof goes by induction with respect to $|C|$.

For $C = \emptyset$, the lemma is trivially true. Let C be such that $|C| = n$ and let $c \in S, c \notin A \cup C$. Let us consider the family

$$(D_\alpha(X \cup Z, A \cup C \cup \{c\}))_{Z \subseteq C \cup \{c\}}.$$

For each $Z \subseteq C \cup \{c\}$, either $c \in Z$ or $c \notin Z$, so either $Z = Y \cup \{c\}$ or $Z = Y$, for some $Y \subseteq C$. Therefore, we can write

$$(D_\alpha(X \cup Z, A \cup C \cup \{c\}))_{Z \subseteq C \cup \{c\}} =$$
$$(D_\alpha(X \cup Y, A \cup C \cup \{c\}), D_\alpha(X \cup Y \cup \{c\}, A \cup C \cup \{c\}))_{Y \subseteq C}.$$

By Proposition 2,

$$D_\alpha(X \cup Y, A \cup C \cup \{c\}) \oplus D_\alpha(X \cup Y \cup \{c\}, A \cup C \cup \{c\}) = D_\alpha(X \cup Y, A \cup C).$$

It only remains to apply the induction hypothesis to finish the proof. □

⟨coro:decomposition⟩ **Corollary 9.** [6] *For every $A \in Fin(S)$, $(D_\alpha(X, A))_{X \subseteq A}$ is a decomposition of unit.*

Proof. By Lemma 2,

$$\bigoplus_{X \subseteq A} (D_\alpha(\emptyset \cup X, \emptyset \cup A)) = D_\alpha(\emptyset, \emptyset) = \alpha(\emptyset) = 1$$

□

[6] This follows directly from the Möbius inversion theorem; how about decomposition lemma itself?

⟨coro:alphaAobservable⟩ **Corollary 10.** *For every $A \in Fin(S)$, the mapping $\omega_A : 2^{(2^A)} \to E$ given by*

$$\omega_A(\mathbb{X}) = \bigoplus_{X \in \mathbb{X}} D_\alpha(X, A)$$

is a simple observable.

Proof. The atoms of $2^{(2^A)}$ are of the form $\{X\}$, where $X \subseteq A$. By Corollary 9, $(\omega_A(\{X\}) : X \subseteq A)$ is a decomposition of unit; the remainder of the proof is trivial. □

For $A, B \in Fin(S)$ with $A \subseteq B$, let us define mappings $g^A_B : 2^{(2^A)} \to 2^{(2^B)}$

$$g^A_B(\mathbb{X}) = \{X \cup C_0 : X \in \mathbb{X} \text{ and } C_0 \subseteq (B \setminus A)\}$$

and let us write $\mathcal{G}$ for the collection of all such mappings.

It is an easy exercise to prove that every $g^A_B \in \mathcal{G}$ is an injective homomorphism of Boolean algebras and that $((2^{(2^A)} : A \in Fin(S)), \mathcal{G})$ is a direct family of Boolean algebras.

Let us prove that the mappings g^A_B behave well with respect to the observables ω_A and ω_B.

⟨lemma:d1commutes⟩ **Lemma 3.** *Let $A, B \in Fin(S)$ with $A \subseteq B$. The diagram*

$$\begin{tikzcd}
2^{(2^A)} \arrow[r, "\omega_A"] \arrow[d, "g^A_B"'] & E \\
2^{(2^B)} \arrow[ur, "\omega_B"']
\end{tikzcd}$$

commutes.

Proof. For all $\mathbb{X} \in 2^{(2^A)}$,

$$\begin{aligned}
\omega_B(g^A_B(\mathbb{X})) &= \omega_B(\{X \cup C_0 : X \in \mathbb{X} \text{ and } C_0 \subseteq (B \setminus A)\}) = \\
&= \bigoplus (D_\alpha(X \cup C_0, B) : X \in \mathbb{X} \text{ and } C_0 \subseteq (B \setminus A)) = \\
&= \bigoplus_{X \in \mathbb{X}} \Big(\bigoplus_{C_0 \subseteq (B \setminus A)} D_\alpha(X \cup C_0, B) \Big)
\end{aligned}$$

Put $Y := C_0$, $C := B \setminus A$; by Lemma 2,

$$\bigoplus_{C_0 \subseteq (B \setminus A)} D_\alpha(X \cup C_0, B) = D_\alpha(X, A).$$

Therefore,

$$\omega_B(g^A_B(\mathbb{X})) = \bigoplus_{X \in \mathbb{X}} D_\alpha(X, A) = \omega_A(\mathbb{X})$$

and the diagram commutes. □

⟨coro:simplerange⟩ **Corollary 11.** *For every $B \in Fin(S)$, B is a subset of the range of ω_B.*

Proof. We need to prove that every $a \in B$ is an element of the range of ω_B. For $B = \emptyset$, this is trivial.

Suppose that B is nonempty and let $a \in B$. Let $A = \{a\}$. and let $X = g_B^A(\{\{a\}\})$. By Lemma 3,

$$\omega_B(X) = \omega_B(g_B^A(\{\{a\}\})) = \omega_A(\{\{a\}\}),$$

and we see that

$$\omega_A(\{\{a\}\}) = \omega_{\{a\}}(\{\{a\}\}) = D_\alpha(\{a\},\{a\}) = \alpha(\{a\}) = a.$$

□

?⟨thm:obsfromcsm⟩? **Theorem 2.** *Let E be an effect algebra. If S admits a compatibility mapping, then S can be embedded into the range of an observable.*

Proof. Suppose that S admits a compatibility mapping. Let us construct $F_B(S)$ as the direct limit of the direct family $(2^{2^A} : A \in Fin(S))$, equipped with morphisms of the type g_B^A. After that, we shall define an observable $\omega : F_B(S) \to E$.

Consider the set

$$\Gamma_S = \bigcup_{A \in Fin(S)} \{(\mathbb{X}, A) : \mathbb{X} \subseteq 2^A\}$$

and define on it a binary relation $\equiv$ by $(\mathbb{X}, A) \equiv (\mathbb{Y}, B)$ if and only if $g_{A\cup B}^A(\mathbb{X}) = g_{A\cup B}^B(\mathbb{Y})$, that means

$$\{X \cup C_A : X \in \mathbb{X} \text{ and } C_A \subseteq A \cup B \setminus A\} = \{Y \cup C_B : Y \in \mathbb{Y} \text{ and } C_B \subseteq A \cup B \setminus B\}.$$

Then $F_B(S) = \Gamma_S/\equiv$ and the operations on $F_B(S)$ are defined by

$$[(\mathbb{X}, A)]_\equiv \vee [(\mathbb{Y}, B)]_\equiv = [(g_{A\cup B}^A(\mathbb{X}) \cup g_{A\cup B}^B(\mathbb{Y}), A \cup B)]_\equiv$$

and similarly for the other operations. Then $F_B(S)$ is a direct limit of Booleat algebras, hence a Boolean algebra.

Let $\omega_S : F_B(S) \to E$ be a mapping given by the rule $\omega_S([(\mathbb{X}, A)]_\equiv) = \omega_A(\mathbb{X})$. We shall prove that ω_S is an observable.

Let us prove ω_S is well-defined. Suppose that $(\mathbb{X}, A) \equiv (\mathbb{Y}, B)$, that means, $g_{A\cup B}^A(\mathbb{X}) = g_{A\cup B}^B(\mathbb{Y})$. By Lemma 3,

$$\omega_A(\mathbb{X}) = \omega_{A\cup B}(g_{A\cup B}^A(\mathbb{X}))$$

and

$$\omega_B(\mathbb{Y}) = \omega_{A\cup B}(g_{A\cup B}^B(\mathbb{Y})),$$

hence ω_S is a well-defined mapping.

Let us prove that ω_S is an observable. The bounds of the Boolean algebra $F_B(S)$ are $[(\emptyset, A)]_\equiv$ and $[(2^A, A)]_\equiv$, where $A \in Fin(S)$. Obviously, by Corollary 10,

$$\omega_S([(\emptyset, A)]_\equiv) = \omega_A(\emptyset) = 0$$

and

$$\omega_S([(2^A, A)]_\equiv) = \omega_A(2^A) = 1.$$

Let $[(\mathbb{X}, A)]_\equiv$ and $[(\mathbb{Y}, B)_\equiv]$ be disjoint elements of $F_B(S)$, that is, $g_{A\cup B}^A(\mathbb{X}) \cap g_{A\cup B}^B(\mathbb{Y}) = \emptyset$. Then

$$\begin{aligned}\omega_S([(\mathbb{X}, A)]_\equiv \vee [(\mathbb{Y}, B)]_\equiv) = \omega_S([g_{A\cup B}^A(\mathbb{X}) \cup g_{A\cup B}^B(\mathbb{Y}), A \cup B]_\equiv) = \\ = \omega_{A\cup B}(g_{A\cup B}^A(\mathbb{X}) \cup g_{A\cup B}^B(\mathbb{Y})).\end{aligned}$$

Since $\omega_{A\cup B}$ is an observable,

$$\omega_{A\cup B}(g_{A\cup B}^A(\mathbb{X}) \cup g_{A\cup B}^B(\mathbb{Y})) = \omega_{A\cup B}(g_{A\cup B}^A(\mathbb{X})) \oplus \omega_{A\cup B}(g_{A\cup B}^B(\mathbb{Y})).$$

It remains to observe that

$$\omega_{A\cup B}(g^A_{A\cup B}(\mathbb{X})) = \omega_S([(\mathbb{X}, A)]_\equiv)$$

and that

$$\omega_{A\cup B}(g^B_{A\cup B}(\mathbb{Y})) = \omega_S([(\mathbb{Y}, B)]_\equiv).$$

Let us prove that the range of ω_S includes S. Let $a \in S$. By Corollary 11, the range of $\omega_{\{a\}}$ includes a and, by an obvious direct limit argument, the range of $\omega_{\{a\}}$ is a subset of the range of ω_S. $\square$