9th Workshop Functional Analysis and its Applications, September 9–14, 2013, Nemecká, Slovakia, 21–26

REGULARIZED OPTIMAL CONTROL PROBLEM FOR A BEAM LYING ON AN ELASTIC FOUNDATION

M. KEČKEMÉTYOVÁ AND I. BOCK

Abstract. We deal with an optimal control problem governed by a nonlinear hyperbolic initial-boundary value problem describing the perpendicular vibrations of a beam lying on an elastic foundation. A variable thickness of a beam plays the role of a control variable. The original equation for the deflection is regularized in order to derive necessary optimality conditions.

1. Introduction

Shape design optimization problems belong to the frequently solved problems with many engineering applications. We deal here with an optimal design problem for an elastic beam vibrating against an elastic foundation. A variable thickness of a beam plays the role of a control variable. The similar problem for the stationary elastic Bernoulli beam is investigated in [8]. We have considered the dynamic state problem in [4]. The equation for the deflections has there the form

$$e(x)u_{tt} + d(e^3(x)u_{xx})_{xx} + q(x)u^+ = f(t,x) \text{ in } (0,T] \times (0,L).$$

In order to derive not only the existence of optimal variable thickness e but also the necessary optimality conditions we regularize the function $u \mapsto u^+$ by

$$u \mapsto g_\delta(u),\ g_\delta(u) = \begin{cases} 0 \ \text{ for } \ u \le 0 \\ \frac{2}{\delta}u^2 - \frac{1}{\delta^2}u^3 \ \text{ for } \ 0 < u < \delta \\ u \ \text{ for } \ u \ge \delta. \end{cases}$$

Solving the state problem we apply the Galerkin method in the same way as in [1], where the rigid obstacle acting against a beam is considered. The compactness method will be used in solving the minimum problem for a cost functional. We apply the approach from [2] in deriving the optimality conditions.

2. Solving of the state problem

2.1. **Setting of the state problem.** We consider a beam of the length $L > 0$. Its variable thickness is expressed by a positive function $x \mapsto e(x),\ x \in [0,L]$, the positive constants $\rho, b,\ E$ are the density of the material, width of the beam and the Young modulus, $\mu \in (0, 0.5)$ is a Poisson ratio and a positive function $x \mapsto q(x),\ x \in [0,L]$ represents the stiffness of the foundation. The beam is clamped on the both ends. Let $F : (0,T] \times (0,L) \mapsto \mathbb{R}$ be a perpendicular load per a unit length acting on the beam. Then the vertical displacement $u : (0,T] \times (0,L) \mapsto \mathbb{R}$ is due to [5] a solution of the following hyperbolic equation

$$\rho b e(x)u_{tt} + \frac{bE}{12(1-\mu^2)}(e^3(x)u_{xx})_{xx} + q_0(x)g_\delta(u) = F(t,x) \text{ in } (0,T] \times (0,L).$$

2010 *Mathematics Subject Classification.* 35L70, 49J20, 74H45, 74K10.

Key words and phrases. Elastic beam, elastic foundation, dynamic problem, regularization, optimal control.

The work presented here was supported by the Ministry of Education of Slovak Republic under VEGA grant 1/0426/12.

Let $u_0,\ v_0 : (0,L) \mapsto \mathbb{R}$ be the initial displacement and velocity and $d = \frac{E}{12\rho(1-\mu^2)}$, $q = \frac{q_0}{\rho b}$, $f = \frac{F}{\rho b}$ be the new material characteristics. Then the vertical displacement $u : (0,T] \times (0,L) \mapsto \mathbb{R}$ solves the hyperbolic initial-boundary value problem

$$e(x)u_{tt} + d(e^3(x)u_{xx})_{xx} + q(x)g_\delta(u) = f(t,x) \text{ in } (0,T] \times (0,L), \tag{1}$$

$$u(t,0) = u(t,L) = u_x(t,0) = u_x(t,L) = 0,\ t \in (0,T], \tag{2}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in (0,L). \tag{3}$$

We introduce the Hilbert spaces

$$H \equiv L_2(0,L),\ H^1(0,L) = \{y \in H :\ y' \in H\},\ H^2(0,L) = \{y \in H^1(0,L) :\ y'' \in H\},$$
$$V \equiv \mathring{H}^2(0,L) = \{y \in H^2(0,L) :\ y(0) = y(L) = y'(0) = y'(L) = 0\}$$

with the inner products and the norms

$$(y,z) = \int_0^L y(x)z(x)\,dx,\ |y|_0 = (y,y)^{1/2},\ y,\, z \in H,$$
$$(y,z)_1 = \int_0^L [y(x)z(x) + y'(x)z'(x)]\,dx,\ \|y\|_1 = (y,y)_1^{1/2},\ y,\, z \in H^1(0,L),$$
$$(y,z)_2 = \int_0^L [y(x)z(x) + y'(x)z'(x) + y''(x)z''(x)]\,dx,\ \|y\|_2 = (y,y)_2^{1/2},\ y,\, z \in H^2(0,L),$$
$$((y,z)) = \int_0^L y''(x)z''(x)\,dx,\ \|y\| = ((y,y))^{1/2},\ y,\, z \in V.$$

We set $I = (0,T),\ Q = I \times (0,L)$. For be a Banach space X we denote by $L_p(I;X)$ the Banach space of all functions $y : I \mapsto X$ such that $\|y(\cdot)\|_X \in L_p(0,T),\ p \geq 1$, by $L_\infty(I;X)$ the space of essentially bounded functions with values in X, by $C(\bar{I};X)$ the space of continuous functions $y : \bar{I} \mapsto X,\ \bar{I} = [0,T]$. For $k \in N$ we denote by $C^k(\bar{I};X)$ the spaces of k-times continuously differentiable functions defined on $\bar{I}$ with values in X. If X is a Hilbert space we set

$$H^k(I;X) = \{v \in C^{k-1}(\bar{I};X) :\ \frac{d^k v}{dt^k} \in L_2(I;X)\}$$

the Hilbert spaces with the inner products

$$(u,v)_{H^k(I,X)} = \int_I [(u,v)_X + \sum_{j=1}^k (u^j, v^j)_X]\,dt,\ k \in N.$$

Further we set $\mathcal{V} = L_\infty(I;V)$ and denote by $\dot{w},\ \ddot{w}$ and $\dddot{w}$ the first, the second and the third time derivative of a function $w : I \to X$. We assume

$$f \in H^1(I;H);\ u_0 \in V \cap H^4(0,L),\ u_0 < 0;\ v_0 \in V,\ q \in C[0,L],\ q > 0;\ e \in W,$$
$$W = \{e \in H^2(0,L),\ 0 < e_{\min} \leq e(x) \leq e_{\max}\ \forall x \in [0,L]\}$$

and formulate a weak solution of the problem (1)-(3).

Definition 2.1. *A function $u \in \mathcal{V}$ is a weak solution of the problem* (1)-(3) *if $\ddot{u} \in L_2(Q)$ and*

$$\int_Q \left[e\ddot{u}y + de^3(x)u_{xx}y_{xx} + q(x)g_\delta(u)y\right]\,dx\,dt = \int_Q f(t,x)y\,dx\,dt\ \forall\, y \in L_2(I;V), \tag{4}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0. \tag{5}$$

2.2. **Existence and uniqueness of the state problem.** We verify the existence and uniqueness of a weak solution by the Galerkin method.

Theorem 2.2. *There exists a unique solution $u \in \mathcal{V}$ of the problem (4,5) such that $\dot{u} \in \mathcal{V} \cap C(\bar{I}; H^{2-\varepsilon}(0,L))$, $\ddot{u} \in L_\infty(I;H)$ $\forall \varepsilon > 0$; fulfil the estimate*

$$\|\ddot{u}\|_{L_\infty(I,H)} + \|\dot{u}\|_{L_\infty(I,V)} \le C(d, e_{\min}, e_{\max}, u_0, v_0, f, q). \tag{6}$$

Proof. Let $\{w_i \in V \cap H^4(0,L);\ i \in \mathbb{N}\}$ be a basis of V. We construct the Galerkin approximation u_m of a solution in a form

$$u_m(t) = \sum_{i=1}^{m} \alpha_i(t) w_i,\ \alpha_i(t) \in \mathbb{R},\ i = 1, ..., m,\ m \in \mathbb{N},$$

$$\int_0^L \left[e(x)\ddot{u}_m(t)w_i + de^3(x)u_{mxx}w_{ixx} + q(x)g_\delta(u_m)w_i\right] dx = \int_0^L f(t)w_i\,dx,\ i = 1, ..., m, \tag{7}$$

$$u_m(0) = u_{0m},\ \dot{u}_m(0) = v_{0m},\ u_{0m} \to u_0 \text{ in } H^4(0,L) \text{ and } v_{0m} \to v_0 \text{ in } V. \tag{8}$$

After applying the theorem on a local existence and uniqueness of a solution $\{\alpha_1, ..., \alpha_m\}$ of the 2nd-order system of ordinary differential equations we obtain the solution u_m which is defined on a certain interval $I_m = (0, t_m),\ t_m < T$. It can be extended to the whole interval $[0,T]$ as a consequence of *a priori* estimates that we prove next.

We multiply the equation (7) with $\dot{\alpha}_i(t)$, sum up with respect to i and integrate. The estimate

$$\|\dot{u}_m\|^2_{C(\bar{I}_m,H)} + \|u_m\|^2_{C(\bar{I}_m,V)} \le C_1(d, e_{\min}, e_{\max}, u_0, v_0, f, q). \tag{9}$$

then follows. As the right-hand side of this estimate does not depend on t_m a solution can be prolonged to the whole interval I with the *a priori* estimate

$$\|\dot{u}_m\|^2_{C(\bar{I},H)} + \|u_m\|^2_{C(\bar{I},V)} \le C_1(d, e_{\min}, e_{\max}, u_0, v_0, f, q). \tag{10}$$

In order to achieve better *a priori* estimates we differentiate (7) with respect to t and insert $\ddot{u}_m$ for w_i. We arrive at

$$\int_0^L \left[e(x)\dddot{u}_m(t)\ddot{u}_m(t) + de^3(x)\dot{u}_{mxx}\ddot{u}_{mxx}\right] dx = \int_0^L \left[\dot{f}(t) - q(x)g'_\delta(u_m)\dot{u}_m\right] \ddot{u}_m(t)\,dx$$

which yields after integrating with respect to the time variable

$$\begin{aligned} &\int_0^L \left[e(x)\ddot{u}_m^2(t) + de^3(x)\dot{u}_{mxx}^2(t)\right] dx = \\ &\int_0^L \left[e(x)\ddot{u}_m^2(0) + de^3(x)v_{0mxx}^2\right] dx + 2\int_0^t \int_0^L \left[\dot{f}(s) - q(x)g'_\delta(u_m)\dot{u}_m\right] \ddot{u}_m(s)\,dx\,ds. \end{aligned} \tag{11}$$

We apply (7) for $t = 0$, set $w_i = \ddot{u}_m(0)$ and obtain

$$\int_0^L e(x)\ddot{u}_m^2(0)\,dx = \int_0^L \left[-d(e^3(x)u_{0mxx})_{xx} + f(0)\right] \ddot{u}_m(0)\,dx. \tag{12}$$

We remark that the assumption $u_0 < 0$ implies $u_{0m}(0) < 0$ and hence $g_\delta(u_{0m}(0)) = 0$ for sufficiently large m. After combining the estimate (10) and the expressions (11,12) we obtain the estimates of acceleration term and velocity term

$$\|\ddot{u}_m\|^2_{C(\bar{I},H)} + \|\dot{u}_m\|^2_{C(\bar{I},V)} \le C_2(d, e_{\min}, e_{\max}, u_0, v_0, f, q),\ m \ge m_0. \tag{13}$$

We proceed with the convergence of the Galerkin approximation. Applying the estimates (10), (13) the Aubin-Lions compact imbedding theorem [6], Sobolev imbedding theorems and the interpolation theorems in Sobolev spaces [3] we obtain for a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$) a function $u \in \mathcal{V}$ with $\dot{u} \in L_\infty(I, V),\ \ddot{u} \in L_\infty(I, H)$ and the convergences

$$
\begin{aligned}
&\ddot{u}_m \rightharpoonup \ddot{u} && \text{in } L_\infty(I, H),\\
&\dot{u}_m \rightharpoonup^* \dot{u} && \text{in } L_\infty(I; V),\\
&\dot{u}_m \to \dot{u} && \text{in } C(\bar{I}; H^{2-\varepsilon}(0, L))\ \forall\, \varepsilon > 0,\\
&\dot{u}_m \to \dot{u} && \text{in } C(\bar{I}; C^1[0, L]),\\
&u_m \rightharpoonup^* u && \text{in } L_\infty(I; V),\\
&u_m \to u && \text{in } C(\bar{I}; H^{2-\varepsilon}(0, L))\ \forall\, \varepsilon > 0,\\
&u_m \to u && \text{in } C(\bar{I}; C^1[0, L]).
\end{aligned}
\tag{14}
$$

Let $\mu \in \mathbb{N},\ y_\mu = \sum_{i=1}^{\mu} \phi_i(t) w_i,\ \phi_i \in \mathcal{D}(0, T),\ i = 1, ..., \mu$. The convergence process (14) implies

$$\int_Q \left[e\ddot{u} y_\mu + de^3(x) u_{xx} y_{\mu xx} + q(x) g_\delta(u) y_\mu \right] dx\, dt = \int_Q f y_\mu\, dx\, dt.$$

Functions $\{y_\mu\}$ form a dense subset of the set $L_2(I; V)$ and hence a function u fulfils the identity (4).

The initial conditions (5) follow due to (8) and the proof of the existence of a solution is complete.

The proof of the uniqueness can be performed in a standard way using the Gronwall lemma.

Remark 2.3. *It is possible after applying the approach from* [7], *11.2.3 to verify the uniqueness in the larger class of solutions with* $\ddot{u} \in L_\infty(I; V^*)$.

3. Optimal control problem

3.1. **The existence of an optimal thickness.** We consider a cost functional $J : L_2(I; V) \times H^2(0, L) \mapsto \mathbb{R}$ fulfilling the assumption

$$u_n \rightharpoonup u \text{ in } L_2(I; V),\ e_n \rightharpoonup e \text{ in } H^2(0, L) \Rightarrow J(u; e) \le \liminf_{n\to\infty} J(u_n; e_n) \tag{15}$$

and

$$E_{ad} = \{e \in H^2(0, L) :\ 0 < e_{\min} \le e(x) \le e_{\max}\ \forall x \in [0, L],\ \|e\|_{H^2(0,L)} \le \hat{e}\}$$

the set of admissible thicknesses. We formulate

Optimal control problem $\mathcal{P}$: To find a control $e^* \in E_{ad}$ such that

$$J(u(e^*), e^*) \le J(u(e), e)\ \forall e \in E_{ad}, \tag{16}$$

where $u(e)$ is a (unique) weak solution of the Problem (1)-(3).

Theorem 3.1. *There exists a solution of the Optimal control problem* $\mathcal{P}$.

Proof. We use the lower semicontinuity properties of the functional J and the compactness of the admissible set E_{ad} of thicknesses in the space $C[0, L]$. Let $\{e_n\} \subset E_{ad}$ be a minimizing sequence for (16) i.e.

$$\lim_{n\to\infty} J(u(e_n); e_n) = \inf_{e \in E_{ad}} J(u(e), e). \tag{17}$$

There exists a subsequence of $\{e_n\}$ (denoted again by $\{e_n\}$) and an element e^* such that

$$e_n \rightharpoonup e^* \text{ in } H^2(0, L),\ e_n \to e^* \text{ in } C[0, L]. \tag{18}$$

The *a priori estimates* (6) imply the existence of a function $u^* \in \mathcal{V} \equiv L_\infty(I; V)$ such that $u_t \in L_\infty(I; H)$ and

$$u(e_n) \rightharpoonup^* u^* \text{ in } \mathcal{V},\ u_t(e_n) \rightharpoonup \dot{u}_t^* \text{in } L_\infty(I; H). \tag{19}$$

Functions $u_n \equiv u(e_n)$ solve the initial value state problem (4-5) for $e \equiv e_n$. The convergences (18-19) then implies that u_* solves the problem (4,5) with $e \equiv e^*$ and $\ddot{u}^* \in L_\infty(I;V^*)$. We have $u^* \equiv u(e^*)$ due to Theorem 2.2 and Remark 2.3. and hence

$$u(e_n) \rightharpoonup^* u(e^*) \text{ in } \mathcal{V},\ u(e_n) \rightharpoonup u(e^*) \text{ in } L_2(I;V).$$

Properties (15) and (17) then imply

$$J(u(e^*), e^*) = \min_{e \in E_{ad}} J(u(e), e)$$

and the proof is complete.

3.2. **Necessary optimality conditions.** In order to derive necessary optimality conditions we assume for simplicity the cost functional in a form

$$J(u,e) = \|\mathcal{C}u - z_d\|_{\mathcal{H}} + N\|e\|_2^2,\ z_d \in \mathcal{H}, N \geq 0,$$

where $\mathcal{H}$ is any Hilbert space, $\mathcal{C} \in \mathcal{L}(L_2(I;V), \mathcal{H})$.

The optimal control problem can then be expressed in a form

$$j(e^*) = \min_{e \in E_{ad}} j(e),\ j(e) = J(u(e), e). \tag{20}$$

Let us introduce the Banach space

$$\mathcal{W} = \{v \in \mathcal{V} : \dot{v} \in \mathcal{V},\ \ddot{v} \in L_\infty(I;H)\}.$$

In the same way as in [2] the following theorem about Fréchet differentiability of the mapping $e \mapsto u(e)$ can be verified.

Theorem 3.2. *The mapping $u(\cdot) : E_{ad} \to \mathcal{W}$ is Fréchet differentiable and its derivative $z \equiv u'(e)h$ fulfils for all $e \in E_{ad}$ the operator equation*

$$\mathcal{A}(e)z = -\mathcal{B}(e)h,\ h \in H^2(0,L), \tag{21}$$

with the operators $\mathcal{A}(e)z$, $\mathcal{B}(e)h : \mathcal{W} \to L_2(I;V^*)$ defined by

$$\langle\langle \mathcal{A}(e)z, y \rangle\rangle = \int_Q \left[e\ddot{z}y + de^3 z_{xx}y_{xx} + q(x)g'_\delta(u(e))zy \right] dx\,dt \tag{22}$$

$$\langle\langle \mathcal{B}(e)h, y \rangle\rangle = \int_Q h[\ddot{u}(e)y + 3de^2 u_{xx}(e)y_{xx}]\,dx\,dt,\ y \in L_2(I;V). \tag{23}$$

The functional j in (20) is Fréchet differentiable and its derivative in $e^* \in E_{ad}$ has the form

$$\langle j'(e^*), h \rangle = 2\langle \mathcal{C}u(e^*) - z_d, \mathcal{C}[u'(e^*)h] \rangle + 2N(e^*, h)_2,\ h \in H^2(0,L). \tag{24}$$

The optimal thickness $e^* \in E_{ad}$ fulfils the variational inequality

$$\langle j'(e^*), e - e^* \rangle \geq 0\ \forall e \in E_{ad} \tag{25}$$

which can be expressed in a form

$$\int_I \langle \mathcal{C}^*\Lambda(\mathcal{C}u(e^*) - z_d), u'(e^*)h \rangle_{V^*,V}\,dt + N(e^*, e - e^*) \geq 0\ \forall e \in E_{ad} \tag{26}$$

with the adjoint operator $\mathcal{C}^* \in \mathcal{L}(\mathcal{H}^*, L_2(I;V^*))$ and the canonical isomorphism $\Lambda : \mathcal{H} \to \mathcal{H}^*$.

Applying Theorem 3.2 we obtain the necessary optimality conditions in a form of a system with an adjoint state p:

Theorem 3.3. *The optimal thickness e^*, the corresponding state (deflection) $u^* \equiv u(e^*)$ and the adjoint state $p^* \equiv p(e^*)$ are solutions of the initial value problem*

$$\int_Q \left[e^* u^*_{tt} y + d(e^*)^3(x) u^*_{txx} y_{xx} + q(x) g_\delta(u^*) y \right] dx\,dt = \int_Q f(t,x) y\,dx\,dt\ \forall\, y \in L_2(I;V),$$

$$u^*(0) = u_0,\ u^*_t(0) = v_0,$$

$$\mathcal{A}(e^*)p^* = \mathcal{C}^*\Lambda(\mathcal{C}u^* - z_d);\ p^*(T) = p^*_t(T) = 0,$$

$$N(e^*, e - e^*)_2 - \langle\langle \mathcal{B}(e^*)(e - e^*), p^* \rangle\rangle\ \forall e \in E_{ad}.$$

Remark 3.4. *It is possible after using the variational inequality* (25) *with* (24) *to obtain for sufficiently large* N *the uniqueness of the Optimal control* e^**.*

References

[1] Bock, I.: An optimization of a Mindlin-Timoshenko beam with a dynamic boundary contact on the boundary. *Proc. Appl. Math. Mech.* **11** (2011), 791–792.

[2] Bock, I., LOVÍŠEK, J.: Optimal control of a viscoelastic plate bending respect to a thickness. *Math. Nachrichten* **125** (1986), 135–151.

[3] Eck C., Jarušek, J. and Krbec, M.: *Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems.* Monographs & Textbooks in Pure & Appl. Math. No. 270 (ISBN 1-57444-629-0). Chapman & Hall/CRC (Taylor & Francis Group), Boca Raton – London – New York – Singapore 2005.

[4] Kečkemétyová, M., Bock, I.: On an optimal control problem for a beam lying on an elastic foundation *Proc. IX. Int. Sci. Conf. of Faculty of Civil Engineering Košice, May 23/25, 2012.*

[5] LAGNESE, J., LEUGERING, G.: Uniform stabilization of a nonlinear beam by nonlinear boundary feedback. *J.Diff. Equations* **16** (1991), 35–54.

[6] Lions, J.L.: *Quelques méthodes de résolutions sep problémes aux limites non linéaires.* Dunod, Paris, 1969.

[7] Renardy, M, Rogers, R.C.: *An Introduction to Partial Differential Equations.* Springer, New York, Berlin, Heidelberg, 2004.

[8] Šimeček, R.: Optimal design of an elastic beam with a unilateral elastic foundation: Semicoercive case. *Applications of Mathematics* **58**, No.3 (2013), 329–346.

Institute of Computer Science and Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava 1, Slovak Republic

E-mail address: maria.keckemetyova@stuba.sk, igor.bock@stuba.sk