Compression of quasianalytic spectral sets of cyclic contractions

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$\mathcal{H}$ complex Hilbert space, $\dim \mathcal{H} = \aleph_0$

$\mathcal{L}(\mathcal{H})$ bounded, linear operators on $\mathcal{H}$

$\mathcal{M} \subset \mathcal{H}$ subspace: closed linear manifold

non-trivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$

invariant for $T \in \mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$

$T \in \mathcal{L}(\mathcal{H})$ is given

Lat T invariant subspace lattice of T

$\{T\}' = \{C \in \mathcal{L}(\mathcal{H}) : CT = TC\}$ commutant of T

Hlat $T = \cap \{\text{Lat } C : C \in \{T\}'\}$

hyperinvariant subspace lattice of T

(ISP) *Does every $T \in \mathcal{L}(\mathcal{H})$ have a non-trivial invariant subspace?*

(HSP) *Does every $T \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}I$ have a non-trivial hyperinvariant subspace?*

H^2 Hardy space of analytic functions on $\mathbb{D}$

$S \in \mathcal{L}(H^2),\ Sh = \chi h,$ where $\chi(z) = z\ \ \forall z \in \mathbb{D}$

unilateral shift; cyclic: $\vee_{n=0}^{\infty} S^n 1 = H^2$

Lat $S =$ Hlat $S = \left\{ \vartheta H^2 : \vartheta \in H^\infty \text{ is inner } \right\}$ (Beurling)

ϑ is inner: $|\vartheta(\zeta)| = 1$ for a.e. $\zeta \in \mathbb{T}$.

Assume $T \in \mathcal{L}(\mathcal{H})$ is a contraction: $\|T\| \leq 1$.

(X, V) is a unitary asymptote of T:

(i) $V \in \mathcal{L}(\mathcal{K})$ is unitary,

(ii) $X \in \mathcal{L}(\mathcal{H}, \mathcal{K}),\ \|X\| \leq 1,\ XT = VX,$

(iii) $\forall (X', V'),\ \exists! Y \in \mathcal{L}(\mathcal{K}, \mathcal{K}'),\ \|Y\| \leq 1,$

$$YV = V'Y,\ X' = YX.$$

Assume $T \in C_{10}$:

(i) $\lim_{n\to\infty} \|T^n x\| > 0 \quad \forall\, 0 \neq x \in \mathcal{H},$

(ii) $\lim_{n\to\infty} \|T^{*n} x\| = 0 \quad \forall x \in \mathcal{H}.$

$\Longrightarrow$ X is injective, and

the unitary V is absolutely continuous (a.c.)

$$(\Lambda(\omega) = 0 \Longrightarrow E(\omega) = 0\ \ \forall\, \omega \subset \mathbb{T})$$

Λ linear measure on $\mathbb{C}$, coinciding with the Lebesgue measure on $\mathbb{T}$ and $\mathbb{R}$

Assume V is cyclic: $\exists\, y \in \mathcal{K},\ \vee_{n=0}^{\infty} V^n y = \mathcal{K}$

$$(\iff \ \exists\, u \in \mathcal{K},\ \vee_{n=-\infty}^{\infty} V^n u = \mathcal{K})$$

$V = M_\alpha$ can be assumed

Here $\alpha \subset \mathbb{T}$ Lebesgue measurable, $L^2(\alpha) = \chi_\alpha L^2(\mathbb{T})$,

$M_\alpha \in \mathcal{L}(L^2(\alpha)),\ M_\alpha f = \chi f$.

$\omega(T) := \alpha$ is the *residual set* of T

$\mathcal{M}_\circ$ Lebesgue measurable subsets of $\mathbb{T}$

$\beta \in \mathcal{M}_\circ$ is quasianalytic for T:

$\forall\, 0 \neq h \in \mathcal{H},\ (Xh)(\zeta) \neq 0$ for a.e. $\zeta \in \beta$.

$b := \sup\{\Lambda(\beta) : \beta \text{ quasianalytic for } T\}$

$\exists\, \{\beta_n\}_{n=1}^{\infty},\ \Lambda(\beta_n) \to b$

$\pi(T) := \cup_n \beta_n$ is the largest quasianalytic set for T

quasianalytic spectral set of T

$$\pi(T) \subset \omega(T)$$

$\pi(T) \neq \omega(T)$ $(\Lambda(\omega(T) \setminus \pi(T)) > 0) \implies$ Hlat T is non-trivial

T is *quasianalytic* if $\pi(T) = \omega(T)$.

$\mathcal{L}_0(\mathcal{H})$ consists of the operators $T \in \mathcal{L}(\mathcal{H})$ satisfying:

(i) T is a C_{10}-contraction,

(ii) V is cyclic,

(iii) T is quasianalytic.

$\mathcal{L}_1(\mathcal{H})$ consists of the operators $T \in \mathcal{L}_0(\mathcal{H})$ satisfying also:

(iv) $\pi(T) = \mathbb{T}$.

$\widetilde{\mathcal{L}}(\mathcal{H})$ consists of the operators $T \in \mathcal{L}(\mathcal{H})$ satisfying:

(i) T is a contraction,

(ii) $\exists\, x \in \mathcal{H},\ \lim_{n\to\infty} \|T^n x\| > 0$,

(iii) V is cyclic.

$$\mathcal{L}_1(\mathcal{H}) \subset \mathcal{L}_0(\mathcal{H}) \subset \widetilde{\mathcal{L}}(\mathcal{H})$$

Positive answer for (HSP) in $\widetilde{\mathcal{L}}(\mathcal{H}) \implies$

pos. answer for (ISP) for every contraction $T \in \mathcal{L}(\mathcal{H})$,

where T or T^* is non-stable

$(\exists\, v \in \mathcal{H},\ \lim_n \|T^n v\| > 0 \ \text{ or } \ \lim_n \|T^{*n} v\| > 0)$

(HSP) in $\widetilde{\mathcal{L}}(\mathcal{H})$ is equivalent to (HSP) in $\mathcal{L}_0(\mathcal{H})$. (LK 2001)

(ISP) is open in $\mathcal{L}_0(\mathcal{H})$

(ISP) has positive answer in $\mathcal{L}_1(\mathcal{H})$:

$\forall\, T \in \mathcal{L}_1(\mathcal{H}),\ \forall\, \varepsilon > 0,\ \vee \mathrm{Lat}_\varepsilon T = \mathcal{H}$, where

$\mathcal{M} \in \mathrm{Lat}_\varepsilon T$ if $\exists\, Q \in \mathcal{L}(\mathcal{M}, H^2),\ \|Q\|\|Q^{-1}\| < 1 + \varepsilon$

and $Q(T|\mathcal{M}) = SQ$. (LK 2007)

(Note $S \in \mathcal{L}_1(H^2)$)

$T \in \mathcal{L}_0(\mathcal{H}),\ \pi(T)$ contains an arc

$\implies \exists\, T_1 \in \mathcal{L}_1(\mathcal{H}),\ \{T_1\}' = \{T\}',\ \mathrm{Hlat}\ T_1 = \mathrm{Hlat}\ T$

(LK 2010)

Theorem 1. $\forall\, T \in \mathcal{L}_0(\mathcal{H}),\ \exists\, T_1 \in \mathcal{L}_1(\mathcal{H}),\ TT_1 = T_1 T.$

$\{T\}'$ and $\{T_1\}'$ are abelian $\implies$

$$\{T_1\}' = \{T\}', \quad \text{Hlat}\ T_1 = \text{Hlat}\ T.$$

Corollary 2.

(HSP) in $\mathcal{L}_0(\mathcal{H})$ is equivalent to (HSP) in $\mathcal{L}_1(\mathcal{H})$.

We want to find T_1 as $T_1 = f(T)$.

$\Phi_T \colon H^\infty \to \mathcal{L}(\mathcal{H}),\ f \mapsto f(T)$ *Sz.-Nagy–Foias functional calculus* for an a.c. contraction $T \in \mathcal{L}(\mathcal{H})$:

contractive, weak-$*$ continuous, algebra-homomorphism,

$\Phi_T(1) = I$ and $\Phi_T(\chi) = T$.

$$\Phi_T(H^\infty) \subset \{T\}'$$

$\Lambda_\circ = \Lambda|\mathcal{M}_\circ$ Lebesgue measure on $\mathbb{T}$

$f \in H^\infty$ *partially inner function*:

(i) $|f(0)| < 1 = \|f\|_\infty$,

(ii) $\Lambda(\Omega(f)) > 0$, where

$$\Omega(f) = \{\zeta \in \mathbb{T} : |f(\zeta) = \lim_{r \to 1-0} f(r\zeta)| = 1\}.$$

$\Omega \subset \Omega(f)$ measurable

$\lambda \colon \mathcal{M}_\circ \to [0, 2\pi]$, $\lambda(\omega) = \Lambda_\circ(f^{-1}(\omega) \cap \Omega)$ a.c. w.r.t. $\Lambda_\circ$

$\text{pe-ran}(f|\Omega) := \{\zeta \in \mathbb{T} : (d\lambda/d\Lambda_\circ)(\zeta) > 0\}$

Spectral Mapping Theorem (LK 2010). *If $T \in \mathcal{L}(\mathcal{H})$ is a quasianalytic a.c. contraction, and $f \in H^\infty$ is a partially inner function, then $f(T)$ is also a quasianalytic contraction, and*

$$\pi(f(T)) = \text{pe-ran}(f|\pi(T, f)), \text{ where } \pi(T, f) = \pi(T) \cap \Omega(f).$$

f is a *regular* partially inner function, if

$f|\Omega(f)$ is *weakly a.c.*:

$$\omega \subset \Omega(f), \Lambda(\omega) = 0 \implies \Lambda(f(\omega)) = 0.$$

$\implies \text{pe-ran}(f|\Omega) = f(\Omega) \quad \forall\, \Omega \subset \Omega(f).$

For $T \in \mathcal{L}_0(\mathcal{H})$ we want to guarantee $f(T) \in \mathcal{L}_0(\mathcal{H})$.

Cyclicity should be preserved $\implies$ univalent functions

A disk algebra: $f\colon \mathbb{D} \to \mathbb{C}$ analytic, and

$$f \text{ can be continuously extended to } \mathbb{D}^-$$

$A_1 = \{f \in A : f|\mathbb{D} \text{ is univalent}\}$

Proposition 3. *$f \in A_1$ partially inner.*

(a) $M = \left\{w \in \mathbb{T} : |f^{-1}(w)|_c > 1\right\}$ *is countable*

$$\implies f|\Omega(f) \textit{ is almost injective.}$$

(b) $\forall\, \Omega \subset \Omega(f)$, $\text{pe-ran}(f|\Omega) = f(\Omega)$

$$\iff f|\Omega \textit{ is weakly a.c..}$$

Theorem 4. *Set* $T \in \mathcal{L}_0(\mathcal{H})$, *and*

$f \in A_1$ *regular partially inner, with* $\Lambda(\pi(T, f)) > 0$.

Then $T_0 = f(T) \in \mathcal{L}_0(\mathcal{H})$ *and* $\pi(T_0) = f(\pi(T, f))$.

$$T \in \mathcal{L}_0(\mathcal{H}) \implies \Lambda(\pi(T)) > 0 \implies$$
$$\exists\, K \subset \pi(T) \text{ compact}, \Lambda(K) > 0$$

Question. *Can we find a regular partially inner function* $f \in A_1$ *such that* $\Omega(f) = K$ *and* $f(K) = \mathbb{T}$*?*

We are looking for an appropriate f in the class of *starlike functions.*

Given

(1) ν *positive Borel measure* on $[0, 2\pi]$,

$\nu([0, 2\pi]) = 2\pi$ and $\nu(\{t\}) = 0 \;\forall\, t \in [0, 2\pi]$,

(2) $\kappa \in \mathbb{C} \setminus \{0\}$.

Consider

$$f(z) = \kappa z \exp\left[-\tfrac{1}{\pi}\int_0^{2\pi} \log(1 - e^{-it}z)\, d\nu(t)\right] \quad (z \in \mathbb{D}).$$

$(\forall\, z \in \mathbb{C} \setminus \mathbb{R}_-,\ \log z := \ln|z| + i \arg z, \text{ where } \arg z \in (-\pi, \pi))$

f is analytic on $\mathbb{D}$, $f(0) = 0$, $f'(0) = \kappa$

$\forall\, z = re^{is} \in \mathbb{D}$:

$$2\pi\ \mathrm{Re}(zf'(z)/f(z)) = \int_0^{2\pi} P_r(s - t)\, d\nu(t) > 0$$

$\Longrightarrow$ f is *starlike*: $f(0) = 0$, f univalent, and

$f(\mathbb{D})$ is starlike $(w \in f(\mathbb{D}) \Longrightarrow [0, w] \subset f(\mathbb{D}))$

$\beta(t) := \nu([0,t])$ $(t \in [0,2\pi])$ *distribution function* of ν

continuous, increasing, $\beta(0) = 0$, $\beta(2\pi) = 2\pi$.

$\kappa := \kappa_0 \exp\left[\frac{i}{2\pi}\int_0^{2\pi} \beta(t)\,dt - i\pi\right]$, where $\kappa_0 \in (0,\infty)$

Then for every $s \in [0,2\pi]$ we have:

$$\lim_{r\to 1-0} \frac{f(re^{is})}{|f(re^{is})|} = \exp[i\beta(s)].$$

$\varphi\colon [0,2\pi] \to \mathbb{T}$, $\varphi(t) = e^{it}$

$\mu(\omega) = \nu(\varphi^{-1}(\omega))/(2\pi)$ *probability Borel measure* on $\mathbb{T}$,

(no atoms)

For every $z \in \mathbb{D}$ we have:

$$|f(z)| = \kappa_0 |z| \exp[-2p_\mu(z)],$$

where

$p_\mu(z) = \int_{\mathbb{T}} \log|z-w|\,d\mu(w)$ is the *potential* of μ.

(p_μ is subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus$ supp μ)

Given $K \subset \mathbb{T}$ **compact**, $0 < \Lambda(K) < 2\pi$.

$\mathcal{P}(K)$ probability Borel measures, with support in K

$\forall\, \eta \in \mathcal{P}(K),\ I(\eta) = \int_K p_\eta(z)\, d\eta(z)$ energy of η

$M(K) = \sup\{I(\eta) : \eta \in \mathcal{P}(K)\} \in \mathbb{R}$

$\exists!\ \mu \in \mathcal{P}(K),\ I(\mu) = M(K)$: *equilibrium measure* of K

(no atoms)

$\mathrm{cap}(K) = \exp[I(\mu)] > 0$ *capacity* of K

Frostman's Theorem:

(i) $p_\mu(z) \geq I(\mu) \quad \forall\, z \in \mathbb{C}$,

(ii) $p_\mu(z) = I(\mu) \quad \forall\, z \in K \setminus F$, where

$F \subset K$ is F_σ with $\mathrm{cap}(F) = 0$,

(iii) $p_\mu(z) > I(\mu) \quad \forall\, z \in \mathbb{C} \setminus K$.

Continuity Principle: $\forall \zeta_0 \in K$,

$p_\mu | K$ is continuous at $\zeta_0 \iff p_\mu$ is continuous at ζ_0.

Wiener's Criterion: $\forall \zeta_0 \in K$, TFAE

(i) $p_\mu(\zeta_0) = I(\mu)$,

(ii) $\sum_{n=1}^{\infty} \frac{n}{\log(2/\text{cap}(K_n))} = \infty$, where

$$K_n = \left\{ \zeta \in K : \gamma^n < |\zeta - \zeta_0| \le \gamma^{n-1} \right\} \quad (\gamma \in (0,1)).$$

Assume $\mathbb{C}_\infty \setminus K$ is a *regular domain*: the previous conditions hold for every $\zeta_0 \in K$.

$\implies p_\mu$ is continuous on $\mathbb{C}$

Define

$$\nu(\omega) = 2\pi\mu(\varphi(\omega)) \quad (\omega \subset [0, 2\pi]), \text{ and}$$

$$\beta(t) = \nu([0, 2\pi]) \quad (t \in [0, 2\pi]).$$

Choose

$$\kappa = (\text{cap}(K))^2 \exp\left[\tfrac{i}{2\pi} \int_0^{2\pi} \beta(t)\, dt - i\pi\right].$$

Consider

$$f(z) = \kappa z \exp\left[-\tfrac{1}{\pi} \int_0^{2\pi} \log(1 - e^{-it} z)\, d\nu(t)\right] \quad (z \in \mathbb{D}).$$

Then $f \in A_1$ and $|f(z)| = 1 \ \forall\, z \in K$.

Suppose the open arc $\widehat{\zeta_1\zeta_2}$ is a component of $\mathbb{T} \setminus K$.

$$\left(\widehat{\zeta_1\zeta_2} = \{e^{it} : t_1 < t < t_2 < t_1 + 2\pi\},\ \zeta_1 = e^{it_1},\ \zeta_2 = e^{it_2}\right)$$

$\mu(\widehat{\zeta_1\zeta_2}) = 0 \implies$

$$\beta(s) = 2\pi\mu(\widehat{1e^{is}}) = \beta(t_1) = \beta(t_2) \ \ \forall\, s \in (t_1, t_2) \implies$$

$$f(\widehat{\zeta_1\zeta_2}) = \{\rho w : r \le \rho < 1\}, \text{ where}$$

$$w = f(\zeta_1) = f(\zeta_2),\ r \in (0,1) \implies$$

$$\Omega(f) = K \ \text{ and } \ f(K) = \mathbb{T}.$$

We know that

$$f(e^{is}) = \exp[i\beta(s)], \ \text{ whenever } \ e^{is} \in K \ (s \in [1, 2\pi]).$$

Hence

$$f|K \ \text{ is weakly a.c.} \iff \beta \ \text{ is a.c.} \iff \mu \ \text{ is a.c..}$$

Proposition 5. *$K \subset \mathbb{T}$ compact, $\Lambda(K) > 0$.*

TFAE

(a) *For the equilibrium measure μ of K we have*

(i) $p_\mu(z) = I(\mu) \quad \forall\, z \in K$,

(ii) μ *is a.c..*

(b) *There exists a regular, partially inner, starlike function $f \in A_1$ such that*

(i) $\Omega(f) = K$,

(ii) $f(K) = \mathbb{T}$.

$\mathcal{C}_+$ is the system of compact sets K on $\mathbb{C}$ such that

(i) $0 < \Lambda(K) < \infty$,

(ii) $\mathbb{C}_\infty \setminus K$ is a regular domain,

(iii) the equilibrium measure μ of K is a.c. (w.r.t. Λ).

Theorem 6. $\forall\, K \subset \mathbb{T}$ *compact*, $\Lambda(K) > 0$, $\forall 0 < \varepsilon < \Lambda(K)$, $\exists\, K_1 \in \mathcal{C}_+$, $K_1 \subset K$ *and* $\Lambda(K \setminus K_1) < \varepsilon$.

The proof of Theorem 6 is reduced to:

Theorem 7. $\forall\, K \subset \mathbb{R}$ *compact*, $\Lambda(K) > 0$, $\forall\, 0 < \varepsilon < \Lambda(K)$, $\exists\, K_1 \in \mathcal{C}_+$, $K_1 \subset K$ *and* $\Lambda(K \setminus K_1) < \varepsilon$.

Main ideas in the proof of Theorem 7:

For $N \in \mathbb{N}$ and $j \in \mathbb{Z}$:

$$I_{N,j} = \left[j2^{-N}, (j+1)2^{-N} \right].$$

For $N \in \mathbb{N}$ and $\varepsilon > 0$:

$$E(N, \varepsilon) = \cup \{ I_{N,j} : j \in \mathbb{Z},\ \Lambda(K \cap I_{N,j}) \geq (1-\varepsilon)\Lambda(I_{N,j}) \}.$$

Lebesgue's Density Theorem $\Longrightarrow$

$$\forall\, \varepsilon > 0, \quad \lim_{N\to\infty} \Lambda(K \cap E(N, \varepsilon)) = \Lambda(K).$$

Given $\varepsilon \in (0, 1/4)$, $\varepsilon_n = \varepsilon/2^n$ $(n \in \mathbb{N})$.

Define N_n $(n \in \mathbb{N})$ by:

$$\Lambda(K \setminus E(N_1, \varepsilon_1)) < \varepsilon_1; \quad N_{n+1} > N_n,$$

$$\Lambda\left((K \cap E(N_n, \varepsilon_n)) \setminus E(N_{n+1}, \varepsilon_{n+1})\right) < \varepsilon_{n+1}/2^{N_n}.$$

Consider

$$E_n := \cap_{k=1}^{n} E(N_k, \varepsilon_k) = \cup_{s=1}^{r_n} [a_{n,s}, b_{n,s}],$$

where $a_{n,1} < b_{n,1} < a_{n,2} < b_{n,2} < \ldots < a_{n,r_n} < b_{n,r_n}$.

The equilibrium measure μ_n of E_n is a.c. (w.r.t. Λ), and for the density function $g_n = d\mu_n/d\Lambda$ we have:

$$g_n(t) = \frac{1}{\pi} \frac{\prod_{s=1}^{r_n - 1} |t - \tau_{n,s}|}{\prod_{s=1}^{r_n} \sqrt{|t - a_{n,s}||t - b_{n,s}|}} \quad (t \in E_n),$$

where $\tau_{n,s} \in (b_{n,s}, a_{n,s+1})$ is the unique solution of

$$\int_{b_{n,s}}^{a_{n,s+1}} \frac{\prod_{s=1}^{r_n - 1} (t - \tau_{n,s})}{\prod_{s=1}^{r_n} \sqrt{|t - a_{n,s}||t - b_{n,s}|}} \, dt = 0 \quad (s = 1, \ldots, r_n - 1).$$

Upper and lower estimates are given for $g_n(t)$.

Then

$$K_1 := \cap_n E_n \text{ is a compact subset of } K,\ \Lambda(K_1) > 0.$$

Wiener's Criterion $\implies$ $\mathbb{C}_\infty \setminus K_1$ is a regular domain.

The equilibrium measure μ of K_1 is a.c. (w.r.t. Λ):

Suppose $\Lambda(\omega) = 0$ for $\omega \subset K_1$; let $\omega' = K_1 \setminus \omega$.

For $I_{N_n,j} \subset E_n$ we have

$$\Lambda(\omega' \cap I_{N_n,j}) = \Lambda(K_1 \cap I_{N_n,j}) \geq (1 - 2\varepsilon_n)\Lambda(I_{N_n,j}),$$

and so

$$\mu(\omega' \cap I_{N_n,j}) \geq \mu_n(\omega' \cap I_{N_n,j}) \geq (1 - c\sqrt{\varepsilon_n})\mu_n(I_{N_n,j}).$$

Summing up for j:

$$\mu(\omega') = \mu(\omega' \cap E_n) \geq (1 - c\sqrt{\varepsilon_n})\mu_n(E_n) = 1 - c\sqrt{\varepsilon_n}.$$

$$\varepsilon_n \to 0 \implies \mu(\omega') = 1 \implies \mu(\omega) = 0.$$

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