Reflexivity and hyperreflexivity for sets of operators

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Proof.

(b) $A \in \mathcal{L}$, $T \in$ Ref($\mathcal{M}$), $\varepsilon > 0$, $x \in \mathscr{X}$;

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- Grivaux, Roginskaya (2008); Müller, Vršovský (2009):

  There exist operators on a Hilbert space which are not orbit-reflexive.

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- *For* $\mathcal{L} \subseteq \mathcal{P}(\mathscr{H})$, *subspace lattice*

  hyperreflexivity $\Longleftrightarrow$ *operator hyperreflexivity*:

  $$d(P,\mathcal{L}) \leq c\alpha(P,\mathscr{L}) \quad (\forall\ P \in \mathcal{P}(\mathscr{H}))?$$

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  - $\dim(\mathscr{X}) = \infty \quad \Rightarrow$
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Corollary

Space of hermitian operators is reflexive.

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Thank you!