Decompositions of contractions and power bounded operators

Vladimir Müller

We will present a survey of results concerning various canonical decompositions of Hilbert space contractions, or more generally, power bounded operators on Banach spaces.

We will discuss the following classical decompositions:

Theorem 1. (mean ergodic theorem) Let X be a reflexive Banach space, let $T \in B(X)$ be a power bounded operator (i.e., $\sup_n \|T^n\| < \infty$). Let $Y_1 = N(T-I)$ and $Z_1 = \overline{R(T-I)}$. Then Y_1, Z_1 are complemented T-invariant subspaces, $X = Y_1 \oplus Z_1$ and $Z_1 = \{x \in X : \lim_{n\to\infty} \|A_n x\| = 0\}$, where $A_n = n^{-1} \sum_{j=0}^{n-1} T^j$.

If T is a Hilbert space contraction, then the spaces Y_1, Z_1 are orthogonal.

Theorem 2. (Jacobs, de Leeuw, Glicksberg) Let X be a reflexive Banach space, let $T \in B(X)$ be a power bounded operator. Let $Y_2 = \bigvee_{|\lambda|=1} N(T-\lambda)$ and $Z_2 = \bigcap_{|\lambda|=1} \overline{R(T-\lambda)}$. Then Y_2, Z_2 are complemented T-invariant subspaces, $X = Y_2 \oplus Z_2$. The subspace Z_2 can be characterized as

$$x \in Z_2 \Leftrightarrow \lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} |\langle T^j x, x^* \rangle| = 0 \text{ for all } x^* \in X^*$$

$$\Leftrightarrow D - \lim_{n\to\infty} \langle T^j x, x^* \rangle = 0 \text{ for all } x^* \in X^*$$

$$\Leftrightarrow \text{ there exists a subsequence } (n_k) \text{ such that } T^{n_k} x \to 0 \text{ weakly,}$$

where $D-\lim a_n = a$ means that there exists a subset $A \subset \mathbb{N}$ of density 1 such that $\lim_{n\in A} a_n = a$.

If T is a Hilbert space contraction, then the spaces Y_2, Z_2 are orthogonal and

$$x \in Z_2 \Leftrightarrow \lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} |\langle T^j x, x \rangle| = 0.$$

Theorem 3. (Fuguel decomposition) Let $T \in B(H)$ be a Hilbert space contraction. Let

$$Z_3 = \{x \in H : T^n x \to 0 \text{ weakly}\}$$

and

$$Y_3 = \bigvee \{x \in H : \text{there is a subsequence } (n_k) \text{ and } y \in H \text{ with } T^{n_k} y \to x \text{ weakly}\}.$$

Then Y_3, Z_3 are T-invariant subspaces and $H = Y_3 \oplus Z_3$ (orthogonal sum).

Theorem 4. (singular/absolutely continuous decomposition) Let $T \in B(H)$ be a Hilbert space contraction. Let Y_4 and Z_4 be the sets of all $x \in H$ such that there exists a singular (absolutely continuous) measure μ_x with

$$\langle p(T)x, x \rangle = \int p \, \mathrm{d}\mu_x$$

for all polynomials p. Then Y_4, Z_4 are orthogonal T-invariant subspaces and $H = Y_3 \oplus Z_3$.

Theorem 5. (unitary/completely non-unitary decomposition) Let $T \in B(H)$ be a Hilbert space contraction. Then there are orthogonal T-invariant subspaces $Y_5, Z_5 \subset H$ such that $T|Y_5$ is unitary and $T|Z_5$ completely non-unitary.

Clearly for any Hilbert space contraction T we have $Y_1 \subset Y_2 \subset Y_3 \subset Y_4 \subset Y_5$ and $Z_1 \supset Z_2 \supset Z_3 \supset Z_4 \supset Z_5$.